ABSTRACT SYMMETRIC BOUNDARY CONDITIONS*

BY

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The topic discussed here belongs to the theory of linear transformations in Hilbert space and presupposes on the part of the reader a fairly thorough knowledge of certain portions of that subject. The following material is at most only slightly more than the minimum prerequisite: M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15, New York, 1932, chaps. 1–5, and chap. 9, §§1, 2, or the various writings of J. von Neumann dealing with the same or related aspects of the theory; J. von Neumann, *Über adjungierte Operatoren*, Annals of Mathematics, (2), vol. 33 (1932), pp. 294–310; F. J. Murray, *Linear transformations between Hilbert spaces and the application of this theory to linear partial differential equations*, these Transactions, vol. 37 (1935), pp. 301–338, §§1–5 only.

I wish here to acknowledge my indebtedness to M. H. Stone who not only suggested the thesis indicated above, but has also made an extremely valuable contribution to the present work. Thanks are due also to J. von Neumann, with whom the author has had several fruitful conversations concerning the theory here developed. More precise acknowledgments are made in the course of the paper.

INTRODUCTION

1. The nature and applications of the subject. The basic concept of the present paper is embodied in a definition (Definition 1.1) which leads to a formula associated with a certain type of transformation T in Hilbert space, analogous to the so-called "fundamental formula" associated with a differential operator coincident with its formal adjoint.† We are thus able to introduce an abstract definition of linear boundary conditions associated with the equation $Tf - \lambda f = g$, and to study the properties of such boundary conditions.

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† We use the term "fundamental formula" in the sense of Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, New Haven, 1923, pp. 58–69; for an ordinary differential operator, the fundamental formula is the familiar Lagrange identity.

In this formulation, the boundary condition is regarded as defining a subset of the domain of T and thus a contraction of the transformation T (that is, a transformation S such that $S \subseteq T$). The main problem which we consider is the determination of those boundary conditions which define self-adjoint and maximal symmetric contractions of T.

The entities introduced in the fundamental definition are realizable in terms of a wide variety of differential operators, both ordinary and partial. In terms of such realizations, the abstract boundary conditions which we consider include the familiar self-adjoint boundary conditions of the classical theory of differential equations. Applied to a differential operator our results serve to characterize a wide class of differential systems for which a unique spectral form exists.

Apart from the introduction, the paper is divided into four chapters. In Chapter I the fundamental definitions are introduced and a few simple but basic theorems established. Here also examples from the field of differential operators are given. The chapter concludes with a precise statement of the important problems to be considered. In Chapter II, manifolds possessing a kind of symmetry, including as special cases manifolds which appear as the graphs of symmetric transformations, are studied in detail. In Chapter III, the situation postulated in Definition 1.1 which gives rise to the "fundamental formula" for an operator T is thoroughly analyzed. In Chapter IV, the results of Chapters II and III are applied to the solution of the problems stated at the end of Chapter I.

A fifth chapter dealing with the applications of the theory to certain types of differential operators was originally planned but is not included; applications will be considered in subsequent papers.

For the convenience of the reader, a detailed table of contents appears at the end of the introduction.

2. **Notation, terminology, and conventions.** Except for minor modification and additions, we use the notation and terminology of M. H. Stone.*

We take occasion here to point out the following notations which we employ systematically and which are not entirely standardized: $\mathfrak{D}$ for a unitary space with dimension number zero and especially for the subspace with dimension number zero of any space under consideration; $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ for the domain and range, respectively, of a transformation T; $T\mathfrak{R}$ for the set in the range of T into which T takes the set $\mathfrak{R}$ in its domain. We reserve the letter E for the designation of projections, and denote a projection with range

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* *Linear Transformations in Hilbert Space and Their Applications to Analysis.* All citations of Stone in the sequel refer to this book.

$\mathfrak{N}$ by $E_{\mathfrak{N}}$. We find it convenient also to adopt the topological notation $\overline{\mathfrak{N}}$ for the closure of the set $\mathfrak{N}$. Since we use this notation only where $\mathfrak{N}$ is a linear manifold, $\overline{\mathfrak{N}}$ is precisely the closed linear manifold determined by $\mathfrak{N}$.

We employ extensively the concept of the graph of a transformation. The graph of a transformation T with domain in a Hilbert space $\mathfrak{H}_1$ and range in a Hilbert space $\mathfrak{H}_2$ is the manifold of vectors $\{f, Tf\}$ in the space $\mathfrak{H}_1 \oplus \mathfrak{H}_2$; in particular, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ may be identical.† We also admit the possibility that either $\mathfrak{H}_1$ or $\mathfrak{H}_2$ is a unitary space. Except where otherwise indicated, we denote the graph of a transformation T by the symbol $\mathfrak{B}(T)$. We recall that $\mathfrak{B}(T)$ is a closed linear manifold if and only if T is a closed linear transformation; and that, if T^* exists, $(\mathfrak{H}_1 \oplus \mathfrak{H}_2) \ominus \mathfrak{B}(T)$ is the linear manifold of vectors $\{T^*f, -f\}$.

At various points we must discuss questions involving a space which may be either a Hilbert space or a unitary space—that is, a separable complex Euclidean space—, and we use the terminology ordinarily associated with the theory of transformations in Hilbert space to cover both cases. This necessitates our taking the definitions of various types of transformations in Hilbert space as definitions of transformations in unitary space also. In many cases the distinctions which these definitions set up for transformations in Hilbert space are vacuous for transformations in a space with finite dimension number. For example, in a unitary space, every linear transformation is bounded, every linear symmetric transformation is self-adjoint, every maximal isometric transformation is unitary. These facts, however, are all well known and in many cases self-evident. We do not, therefore, make explicit comment at every point in the sequel where specialization to the case of unitary space makes modification of the exposition possible.

Although we have occasion to discuss mathematical relations involving several inner products, not all formed in the same space, we use the same notation, namely (,), for all inner products under consideration and state in which space each is formed only when the context fails to make it clear.

In discussing the orthogonal sum, $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n$, of a finite collection of spaces, we find it convenient to use the notation $\mathfrak{H}_k$ to mean, as well as the space $\mathfrak{H}_k$ itself, that manifold of vectors in $\mathfrak{H}$ whose components in $\mathfrak{H}_j$, $(j \neq k)$, are all zero. More generally, if $\mathfrak{M}$ denotes a manifold in $\mathfrak{H}_{k_1} \oplus \mathfrak{H}_{k_2} \oplus \cdots \oplus \mathfrak{H}_{k_m}$, $(m < n)$, we shall on occasion use the symbol $\mathfrak{M}$ also to mean that manifold in $\mathfrak{H}$ whose projection on $\mathfrak{H}_{k_1} \oplus \mathfrak{H}_{k_2} \oplus \cdots \oplus \mathfrak{H}_{k_m}$ is $\mathfrak{M}$

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† The notion of the graph for the case $\mathfrak{H}_1 = \mathfrak{H}_2$ is due to J. von Neumann, Annals of Mathematics, (2), vol. 33 (1932), pp. 294–310; especially p. 299. The more general definition was introduced by F. J. Murray, these Transactions, vol. 37 (1935), pp. 301–338; especially pp. 302–303. All future citations of Murray refer to this paper.

and whose projection on $\mathfrak{H}_j$, $(j \neq k_1, k_2, \cdots, k_m)$, is zero. However, when there is danger of ambiguity, we use a different convention. Thus, if the space under consideration is $\mathfrak{H} \oplus \mathfrak{H}$ and $\mathfrak{M}$ is a manifold in $\mathfrak{H}$, we shall use the notation $\mathfrak{M} + \mathfrak{O}$ to denote the manifold of vectors $\{f, 0\}$ in $\mathfrak{H} \oplus \mathfrak{H}$ such that f is in $\mathfrak{M}$.

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CHAPTER I. FUNDAMENTAL CONCEPTS

1. Reduction operators. The basic notion of this paper, indicated roughly in the introduction, we now state precisely.

DEFINITION 1.1. *Let H be a closed linear transformation in a Hilbert space $\mathfrak{H}$, and let H^* exist. A transformation A with domain in the graph of H^* and*

with range in a unitary or Hilbert space $\mathfrak{M}$ is said to be a reduction operator for H^* if the following conditions are satisfied:

(1) A is closed, linear, and has domain dense in $\mathfrak{B}(H^*)$;

(2) there exists a unitary transformation W in $\mathfrak{M}$ such that

$$(\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}) \ominus \mathfrak{B}(A)$$

is the set of all vectors $\{H^*f, -f, WA\{f, H^*f\}\}$.

The space $\mathfrak{M}$ is called the range-space of A.†

It may be observed that Definition 1.1 can be modified as follows: The condition that A be closed linear can be omitted from (1) and the condition (2) can be stated in a form which does not require $\mathfrak{B}(A)$ to be a closed linear manifold. It can then be proved that A is necessarily closed and linear.

From Definition 1.1, we have at once the formula

$$(1.1) \qquad (f, H^*g) - (H^*f, g) + (A\{f, H^*f\}, WA\{g, H^*g\}) = 0,$$

for all f and g in $\mathfrak{B}(H^*)$ such that $A\{f, H^*f\}$ and $A\{g, H^*g\}$ are defined. In order to simplify the notation, we shall hereafter often write Af for $A\{f, H^*f\}$. Thus the abstract "Lagrange identity" (1.1) may be written

$$(1.2) \qquad (f, H^*g) - (H^*f, g) = - (Af, WAg).$$

THEOREM 1.1. *The transformation H is symmetric. The domain of A contains the graph of H, and $A\{f, H^*f\} = 0$ if and only if f is in the domain of H. Thus $\mathfrak{D}(A) = \mathfrak{B}(H)$ if and only if H is self-adjoint.*

Since H^* exists and H is linear, H has domain dense in $\mathfrak{H}$.‡ Moreover, since H is closed, H^* has domain dense in $\mathfrak{H}$, and H^{**} exists and is identically H.§

If $A\{f, H^*f\} = 0$, then $\{H^*f, -f, 0\}$ is in $(\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}) \ominus \mathfrak{B}(A)$, by Definition 1.1, (2). Thus $(g, H^*f) - (H^*g, f) = 0$ for all $\{g, H^*g\}$ in $\mathfrak{D}(A)$. Since $\mathfrak{D}(A)$ is dense in $\mathfrak{B}(H^*)$, it follows from the identity $H \equiv H^{**}$ that f is in $\mathfrak{D}(H)$ and that $H^*f = Hf$. On the other hand, if f is in $\mathfrak{D}(H)$ it follows, again from the identity $H \equiv H^{**}$, that $\{Hf, -f, 0\}$ is in $(\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}) \ominus \mathfrak{B}(A)$. Therefore $\{f, Hf\}$ is in $\mathfrak{D}(A)$ and $A\{f, Hf\} = 0$. Thus $\mathfrak{D}(A) \supseteqq \mathfrak{B}(H)$ and $A\mathfrak{B}(H) = \mathfrak{O}$. Moreover, since $\mathfrak{B}(H^*) \supseteqq \mathfrak{D}(A)$, we have $\mathfrak{B}(H^*) \supseteqq \mathfrak{B}(H)$. Hence $H^* \supseteqq H$, and, since $\mathfrak{D}(H)$ determines $\mathfrak{H}$, H is symmetric.

Finally, since $\mathfrak{D}(A)$ is dense in $\mathfrak{B}(H^*)$ and H is closed, the equations

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† Compare our previous definition, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 38–42, Definition 1. That the requirement $W^2 + I \equiv 0$ is unnecessary was pointed out to us by J. von Neumann (cf. Theorem 1.2 below).

‡ Stone, Theorem 2.6.

§ Von Neumann, loc. cit., Theorem 2.

$\mathfrak{D}(A) =\mathfrak{B}(H)$ and $\mathfrak{B}(H^*) =\mathfrak{B}(H)$ are clearly coextensive. Hence the identities $A \equiv 0$ and $H^* \equiv H$ are also.

Theorem 1.2. *The range of A is dense in $\mathfrak{M}$, and W satisfies the identity $W^2+I \equiv 0$.*

Since A is linear, $\mathfrak{R}(A)$ is a linear manifold and is therefore dense in $\mathfrak{M}$ if and only if $\mathfrak{M}\ominus\overline{\mathfrak{R}}(A) =\mathfrak{O}$. To establish the latter identity we have only to observe that if k belongs to $\mathfrak{M}\ominus\overline{\mathfrak{R}}(A)$, then

$$(f, 0) - (H^*f, 0) + (Af, k) = 0$$

for all $\{f, H^*f\}$ in $\mathfrak{D}(A)$, whence it follows that $k = WA\{0, 0\} =0$.

To prove the second assertion of the theorem we set $g =f$ in the identity (1.2) to obtain

$$(f, H^*f) - (H^*f, f) = - (Af, WAf).$$

Taking the complex conjugate of both members of this equation, we have

$$(H^*f, f) - (f, H^*f) = - (WAf, Af).$$

Thus, for h in $\mathfrak{R}(A)$, we obtain by addition of the two preceding equations the relation $(Wh, h)+(h, Wh)=0$. Moreover, since W is unitary, $(h, Wh) =(W^{-1}h, h)$ and thus $(Wh, h)+(W^{-1}h, h)=0$ for all h in $\mathfrak{R}(A)$. Hence, since W and W^{-1} are bounded and $\overline{\mathfrak{R}}(A) =\mathfrak{M}$, we have $(Wh+W^{-1}h, h) =0$ for all h in $\mathfrak{M}$. But $W+W^{-1}$ is self-adjoint; therefore the result just obtained implies that its bound is zero. Consequently $W+W^{-1}\equiv O$† or $W^2+I \equiv O$, as we wished to prove.

Theorem 1.3. *Let A be a reduction operator for H^*, and let T be an arbitrary bounded self-adjoint transformation in $\mathfrak{H}$. Let C be the transformation which has as its domain the set of elements $\{f, (H^*+T)f\}$ of $\mathfrak{B}(H^*+T)$ such that $\{f, H^*f\}$ is in $\mathfrak{D}(A)$, and which takes $\{f, (H^*+T)f\}$ into $A\{f, H^*f\}$. Then C is a reduction operator for H^*+T.*

To prove that C is a reduction operator for H^*+T, we seek all elements $\{g^*, g, h\}$ of $\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{M}$ such that

$$(f, g^*) - ((H^* + T)f, g) + (Af, h) = 0$$

for all $\{f, H^*f\}$ in $\mathfrak{D}(A)$. Since $\mathfrak{D}(A) \supseteq\mathfrak{B}(H)$ and $A\mathfrak{B}(H)=\mathfrak{O}$, the identity $(H+T)^* \equiv H^*+T$, which holds by virtue of the fact that T is bounded, implies that g is in $\mathfrak{D}(H^*) =\mathfrak{D}(H^*+T)$ and that $g^* =(H^*+T)g$. Hence, since T is self-adjoint, $\{g^*, g, h\}$ satisfies the above equation for all $\{f, H^*f\}$ in $\mathfrak{D}(A)$ if and only if

† Stone, Theorem 2.22.

$$(f, H^*g) - (H^*f, g) + (Af, h) = 0$$

for all such $\{f, H^*f\}$. But the latter condition is satisfied if and only if $h = WAg$. Thus, in view of the remark following Definition 1.1 we conclude that C is a reduction operator for H^*+T.

To reveal more clearly the significance of Definition 1.1, we give now some concrete examples of reduction operators.

EXAMPLE 1. $\mathfrak{H}$ is the space $\mathfrak{L}_2(a, b)$ where $a \leq x \leq b$ is a finite interval of the real axis. H^* is the transformation which takes f into if' and has for its domain $\mathfrak{D}^*$ the set of all elements f of $\mathfrak{L}_2(a, b)$ which are absolutely continuous on $a \leq x \leq b$ and such that $\int_a^b |f'|^2 dx < \infty$. The operator H is the contraction of H^* whose domain $\mathfrak{D}$ is the set of all elements in $\mathfrak{D}^*$ which vanish at a and b. The space $\mathfrak{M}$ is a two-dimensional unitary space, and A is the transformation which takes $\{f, H^*f\}$ into the point of $\mathfrak{M}$ with coordinates $(f(b), f(a))$; W is the transformation which takes the point (c, d) in $\mathfrak{M}$ into $(ic, -id)$. Here the formula (1.1) is the formula of Lagrange,

$$\int_a^b f\overline{ig'}dx - \int_a^b if'\overline{g}dx + f(b)\overline{ig(b)} - f(a)\overline{ig(a)} = 0.$$

EXAMPLE 2. $\mathfrak{H}$ is the same as in Example 1; the domain $\mathfrak{D}^*$ of H^* is the set of all elements f of $\mathfrak{L}_2(a, b)$ such that f and f' are absolutely continuous on $a \leq x \leq b$ and $\int_a^b |f''|^2 dx$ is finite; $H^*f = f''$. The domain $\mathfrak{D}$ of H consists of those and only those elements g of $\mathfrak{D}^*$ such that $g(a) = g(b) = g'(a) = g'(b) = 0$; $Hg = g''$. The space $\mathfrak{M}$ is a four-dimensional unitary space;

$$A\{f, H^*f\} = \{f(b), f(a), f'(a), -f'(b)\}.$$

The transformation W takes (h, k, l, m) into $(m, l, -k, -h)$ and the formula (1.1) is

$$\int_a^b f\overline{g''}dx - \int_a^b \overline{g}f''dx - f(b)\overline{g'(b)} + f(a)\overline{g'(a)} - f'(a)\overline{g(a)} + f'(b)\overline{g(b)} = 0.$$

EXAMPLE 3. $\mathfrak{H}$ is the space $\mathfrak{L}_2(E)$ where E is the set in the (x, y)-plane bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$, $a < b$, $c < d$. The domain $\mathfrak{D}^*$ of H^* is the set of all elements $f(x, y)$ of $\mathfrak{L}_2(E)$ which are absolutely continuous on $a \leq x \leq b$ for almost all y on $c < y < d$ and for which $\int_E |f_x|^2 dE$ is finite; $H^*f = if_x$. The domain $\mathfrak{D}$ of H is the set of all elements g of $\mathfrak{D}^*$ such that $\lim_{x \to a} g(x, y) = \lim_{x \to b} g(x, y) = 0$ for almost all y on $c < y < d$; $Hg = ig_x$. The space $\mathfrak{M}$ is the space $\mathfrak{L}_2(c, d) \oplus \mathfrak{L}_2(c, d)$;

$$Af = \{f(b, y), f(a, y)\}; \qquad W\{h(y), k(y)\} = -i\{h(y), -k(y)\}.$$

Here the formula (1.1) is

$$\int_E f\overline{ig_x}dE + \int_E if_x\overline{g}dE - \int_c^d f(b, y)\overline{ig(b, y)}dy + \int_c^d f(a, y)\overline{ig(a, y)}dy = 0,$$

and is valid for all f and g in $\mathfrak{D}^*$.

EXAMPLE 4. $\mathfrak{H}$ is the space $\mathfrak{L}_2(E)$ where E is the interior of the unit circle in the (x, y)-plane. The domain $\mathfrak{D}^*$ of H^* is the set of all elements f of $\mathfrak{L}_2(E)$ with the following properties:

(a) f, f_x are absolutely continuous in x on the closed intervals $a \leq x \leq b$, $y = k$ interior to E for almost all values of k on $-1 < y < 1$; f, f_y are absolutely continuous in y on the closed intervals $c \leq y \leq d$, $x = h$, interior to E for almost all values of h on $-1 < x < 1$;

(b) The integrals

$$\int_S |f_x|^2 dS, \quad \int_S |f_y|^2 dS, \quad \int_S |f_{xx}|^2 dS, \quad \int_S |f_{yy}|^2 dS$$

exist on every closed set S interior to E;

(c) $-\nabla^2 f = -f_{xx} - f_{yy}$ is an element of $\mathfrak{L}_2(E)$.
H^* is the transformation with domain $\mathfrak{D}^*$ which takes f into $-\nabla^2 f$. The domain $\mathfrak{D}$ of H is the subset of elements g of $\mathfrak{D}^*$ such that $g = g_x = g_y = 0$ almost everywhere on the boundary C of E. The space $\mathfrak{M}$ is $\mathfrak{L}_2(C) \oplus \mathfrak{L}_2(C)$. The domain of A is the set of all elements $\{f, H^*f\}$ of $\mathfrak{B}(H^*)$ for which $f(s)$ and $\partial f/\partial n = -f_x y'(s) + f_y x'(s)$, where s denotes arc length on C, are elements of $\mathfrak{L}_2(C)$;

$$Af = \{f(s), -\partial f/\partial n\}.$$

W is the transformation which takes $\{h(s), k(s)\}$ into $\{k(s), -h(s)\}$. The formula (1.1) is here the familiar identity of Green,

$$-\int_E f\overline{\nabla^2 g}dE + \int_E \overline{g}\nabla^2 f dE - \int_C f\overline{\partial g/\partial n} \cdot ds + \int_C \partial f/\partial n \cdot \overline{g}ds = 0.$$

Here the domain of A is not identically $\mathfrak{B}(H^*)$ as it is in each of the first three examples.

In later papers we shall deal in some detail with applications of the theory developed in the present memoir. Consequently we omit here proofs that the examples just given are valid illustrations of the situation described in Definition 1.1.

By means of Theorem 1.3, further examples are readily constructed from the ones just given. In particular; in Theorem 1.3 if H^* is a differential operator and T an integral operator, the sum $H^* + T$ is an integro-differential oper-

ator of a type previously studied from various points of view by many writers.

Before proceeding, we point out that it is possible to define a reduction operator A on the graph of the adjoint of an arbitrary symmetric transformation H. A proof of this fact is given later (Theorem 2.9).

2. **Linear boundary conditions.** We introduce now a general definition of linear boundary conditions which describes, in particular, all the linear self-adjoint boundary conditions ordinarily considered in connection with the differential operators of Examples 1–4.

DEFINITION 1.2. *Let A be a reduction operator for H^* with range-space $\mathfrak{M}$. Let $\mathfrak{N}$ be a linear manifold in $\mathfrak{M}$, and let $\mathfrak{D}(\mathfrak{N})$ be the set of elements f in the domain of H^* such that Af is defined and belongs to $\mathfrak{N}$. Then $H(\mathfrak{N})$ denotes the contraction of H^* with domain $\mathfrak{D}(\mathfrak{N})$ and $Af \, \varepsilon \, \mathfrak{N}$ is called the boundary condition defining $H(\mathfrak{N})$. If the equation $Af = h$ does not have a solution f for a dense set of elements h in $\mathfrak{N}$, the boundary condition is said to be degenerate; otherwise it is said to be nondegenerate.*

THEOREM 1.4. *The transformation $H(\mathfrak{N})$ of Definition 1.2 is a linear extension of H and $H^* \supseteq H^*(\mathfrak{N}) \supseteq H(\mathfrak{M} \ominus W \mathfrak{N})$, where W has the same meaning as in Definition 1.1. If T is an arbitrary linear extension of H such that $\mathfrak{B}(T) \subseteq \mathfrak{D}(A)$, then $\mathfrak{N} = A \mathfrak{B}(T)$ is a linear manifold in $\mathfrak{M}$ and $T \equiv H(\mathfrak{N})$.*

Since $\mathfrak{N}$ is a linear manifold and A a linear transformation, the condition $Af \, \varepsilon \, \mathfrak{N}$ clearly describes a linear manifold of elements f of $\mathfrak{D}(H^*)$. Thus, since H^* is linear, $H(\mathfrak{N})$ is also. Furthermore, since $\mathfrak{N}$ is linear, it contains the element 0 of $\mathfrak{M}$. Therefore, by Theorem 1.1, $H(\mathfrak{N}) \supseteq H$, and this implies $H^* \supseteq H^*(\mathfrak{N})$. The relation $H^*(\mathfrak{N}) \supseteq H(\mathfrak{M} \ominus W \mathfrak{N})$ is an immediate consequence of Definition 1.2 and the identity (1.2).

If $\mathfrak{B}(T) \subseteq \mathfrak{D}(A)$, it follows at once, from the fact that $\mathfrak{B}(T)$ is a linear manifold and A a linear transformation, that $\mathfrak{N} = A \mathfrak{B}(T)$ is a linear manifold. The relation $T \equiv H(\mathfrak{N})$ is a consequence of Definition 1.2 and the hypothesis $\mathfrak{B}(T) \subseteq \mathfrak{D}(A)$.

3. **The fundamental problem.** We are primarily concerned here with those transformations $H(\mathfrak{N})$ which are symmetric. In order to isolate the boundary conditions $Af \, \varepsilon \, \mathfrak{N}$ which define such extensions of H, we introduce the following definition:

DEFINITION 1.3. *If $\mathfrak{M}$ is a unitary or Hilbert space and W is a unitary transformation in $\mathfrak{M}$ such that $W^2 + I \equiv 0$, a linear manifold $\mathfrak{N}$ in $\mathfrak{M}$ is said to be W-symmetric if $W \mathfrak{N} \subseteq \mathfrak{M} \ominus \mathfrak{N}$.*

THEOREM 1.5. *If $\mathfrak{N}$ is a linear W-symmetric manifold, the transformation $H(\mathfrak{N})$ of Definition 1.2 is a linear symmetric extension of H. If S is a linear*

symmetric extension of H such that $\mathfrak{B}(S) \subsetneqq \mathfrak{D}(A)$, *then* $A\mathfrak{B}(S)$ *is a linear W-symmetric manifold in* $\mathfrak{M}$.

Theorem 1.5 follows from Theorem 1.4 and the identity (1.2).

We can now state with some precision the twofold problem whose solution is our primary object.

PROBLEM. (1) *To determine conditions on* $\mathfrak{N}$ *necessary and sufficient for* $H(\mathfrak{N})$ *to be maximal symmetric and conditions on* $\mathfrak{N}$ *necessary and sufficient for* $H(\mathfrak{N})$ *to be self-adjoint;* (2) *if* $\mathfrak{D}(A) \neq \mathfrak{B}(H^*)$, *to determine conditions on* $\mathfrak{N}$ *necessary and sufficient for* $\tilde{H}(\mathfrak{N})$ *to be maximal symmetric and conditions necessary and sufficient for* $\tilde{H}(\mathfrak{N})$ *to be self-adjoint.*

We leave to the reader the interpretation in terms of Examples 1–4, and in terms of others which he may construct, of the concepts introduced in this section and the one preceding.†

CHAPTER II. *W*-SYMMETRIC MANIFOLDS

1. **Isometric transformations.** The present chapter is devoted almost entirely to an analysis of the concept of.*W*-symmetry introduced in Definition 1.3. This analysis is based on a simple correspondence between *W*-symmetric manifolds and isometric transformations which is immediately suggested by a well known correspondence between symmetric and isometric transformations,‡ and which has previously been described by O. Teichmüller.§ Before proceeding to the discussion of this correspondence, we state in a form adapted to our special purposes certain facts concerning isometric transformations.

DEFINITION 2.1. *Let* V *be a closed isometric transformation from a space* $\mathfrak{H}_1$ *to a space* $\mathfrak{H}_2$ *where each of the spaces* $\mathfrak{H}_1$, $\mathfrak{H}_2$ *is either a unitary space or a Hilbert space. Let m and n be, respectively, the dimension numbers of the manifolds* $\mathfrak{H}_1 \ominus \mathfrak{D}(V)$ *and* $\mathfrak{H}_2 \ominus \mathfrak{R}(V)$. *Then the number pair* (m, n) *is called the* $(\mathfrak{H}_1, \mathfrak{H}_2)$-*deficiency index of* V. *If* V *is a non-closed isometric transformation from* $\mathfrak{H}_1$ *to*

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† In this connection, the following material will be found suggestive: chap. 10, §§2, 3 of the book of Stone, especially Theorems 10.7, 10.16, and 10.18; the paper of I. Halperin, *Closures and adjoints of linear differential operators*, Annals of Mathematics, (2), vol. 38 (1937), pp. 880–919; especially pp. 883–891; the writer's abstract 43-3-114, Bulletin of the American Mathematical Society.

‡ J. von Neumann, Mathematische Annalen, vol. 102 (1929), pp. 49–131, especially pp. 80–91; or Stone, chap. 9.

§ *Operatoren im Wachsen Raum*, Journal für die reine und angewandte Mathematik (Crelle), vol. 174 (1935), pp. 73–124; especially pp. 99–107. Some of the theorems of the present chapter are only slight variations of results stated by Teichmüller. However, since his analysis does not lend itself readily to our purposes and would, in any case, have to be considerably supplemented at several points, it appeared to us desirable to carry through a complete independent treatment.

$\mathfrak{H}_2$, *the* $(\mathfrak{H}_1, \mathfrak{H}_2)$-*deficiency index of* $\tilde{V}$ *is also said to be the* $(\mathfrak{H}_1, \mathfrak{H}_2)$-*deficiency index of* V. *If* V *has either domain identically* $\mathfrak{H}_1$ *or range identically* $\mathfrak{H}_2$, V *is said to be a maximal isometric transformation from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$; *if both conditions are satisfied,* V *is said to be a unitary transformation from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$.

THEOREM 2.1. *Let* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$ *be separable complex euclidean spaces with dimension numbers* j *and* k, *respectively. A necessary and sufficient condition that a closed isometric transformation from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$ *be unitary (maximal isometric) is that its* $(\mathfrak{H}_1, \mathfrak{H}_2)$-*deficiency index be* $(0, 0)$ *(either* $(p, 0)$ *or* $(0, p)$*).*

If either j *or* k *is zero, the class of all isometric transformations from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$ *contains only the linear transformation whose domain is* $\mathfrak{O}$. *If both* j *and* k *are different from zero, the class of all maximal isometric transformations from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$ *has the cardinal number of the continuum.*

Let j *and* k *be different from zero. If* $j = k < \aleph_0$, *every maximal isometric transformation from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$ *is unitary. If* $j \neq k$, *the* $(\mathfrak{H}_1, \mathfrak{H}_2)$-*deficiency index of every maximal isometric transformation from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$ *is* $(j - k, 0)$ *or* $(0, k - j)$ *according as* j *is greater or less than* k. *If* j *is greater (less) than* k, *and* $\mathfrak{M}$ *is an arbitrary closed linear manifold in* $\mathfrak{H}_1$ $(\mathfrak{H}_2)$ *such that* $\mathfrak{H}_1 \ominus \mathfrak{M}$ $(\mathfrak{H}_2 \ominus \mathfrak{M})$ *has the dimension number* k (j), *then every closed isometric transformation with domain* $\mathfrak{H}_1 \ominus \mathfrak{M}$ *and range* $\mathfrak{H}_2$ *(domain* $\mathfrak{H}_1$ *and range* $\mathfrak{H}_2 \ominus \mathfrak{M}$*) is contained in the class of all maximal isometric transformations from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$. *The class of all such transformations has the cardinal number of the continuum.*

If $j = k = \aleph_0$ *and* p *is an arbitrary cardinal number on the range* $0 \leq p \leq \aleph_0$, *the set of all isometric transformations from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$ *with the* $(\mathfrak{H}_1, \mathfrak{H}_2)$-*deficiency index* $(0, p)$ $((p, 0))$ *has the cardinal number of the continuum. If* $\mathfrak{M}$ *is an arbitrary closed linear manifold in* $\mathfrak{H}_1$ $(\mathfrak{H}_2)$ *whose orthogonal complement has the dimension number* $\aleph_0$, *then every closed isometric transformation with domain* $\mathfrak{H}_1 \ominus \mathfrak{M}$ *and range* $\mathfrak{H}_2$ *(domain* $\mathfrak{H}_1$ *and range* $\mathfrak{H}_2 \ominus \mathfrak{M}$*) belongs to the class of all maximal isometric transformations from* $\mathfrak{H}_1$ *to* $\mathfrak{H}_2$. *The class of all such transformations has the cardinal number of the continuum.*

The assertions of the preceding theorem which are not obvious are either direct consequences of theorems of Stone (op. cit., chap. 2, §5) or are easily proved by methods similar to those used there. We therefore omit proof.

2. *W*-**symmetric manifolds and isometric transformations.** We state first the following definition:

DEFINITION 2.2. *A* W-*symmetric manifold* $\mathfrak{N}$ *in a space* $\mathfrak{M}$ *is said to be maximal* W-*symmetric if it has no proper* W-*symmetric extension. It is said to be hypermaximal* W-*symmetric if* $W\mathfrak{N} = \mathfrak{M} \ominus \mathfrak{N}$.

We come now to the fundamental theorem on *W*-symmetric manifolds.

THEOREM 2.2. *Let W be a unitary transformation in a unitary or Hilbert space $\mathfrak{M}$, and let W satisfy the identity $W^2 + I \equiv O$. Let $\mathfrak{M}^+$ and $\mathfrak{M}^-$ be, respectively, the characteristic manifolds of W for the characteristic values $+i$ and $-i$. Let $\mathcal{U}_W$ be the class of all isometric transformations from $\mathfrak{M}^+$ to $\mathfrak{M}^-$, and let S_W be the class of all linear W-symmetric manifolds $\mathfrak{N}$ in $\mathfrak{M}$. Then there is a one-to-one correspondence between the classes $\mathcal{U}_W$ and S_W such that, if V and $\mathfrak{N}$ correspond, $\mathfrak{N}$ is the range of $I - V$. If V and $\mathfrak{N}$ correspond, so also do $\tilde{V}$ and $\tilde{\mathfrak{N}}$; $\mathfrak{N}$ is a closed linear manifold if and only if V is a closed transformation. The correspondence between $\mathcal{U}_W$ and S_W is an isomorphism with respect to the relation $\subset$. If $\mathfrak{N}$ is a closed linear manifold in S_W and V its correspondent in $\mathcal{U}_W$, then*

$$(2.1) \qquad \mathfrak{M} \ominus \mathfrak{N} = W\mathfrak{N} + (\mathfrak{M}^+ \ominus \mathfrak{D}(V)) + (\mathfrak{M}^- \ominus \mathfrak{R}(V)), \dagger$$

the three component manifolds in the right-hand member being mutually orthogonal. A manifold $\mathfrak{N}$ in S_W is maximal (hypermaximal) W-symmetric if and only if the correspondent V in $\mathcal{U}_W$ is a maximal isometric (unitary) transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$.

We observe first that $\mathfrak{M}^- = \mathfrak{M} \ominus \mathfrak{M}^+$. This follows from the unitary character of W and the easily established fact that the spectrum of W consists only of the two points $+i$ and $-i$.

Consider now an arbitrary manifold $\mathfrak{N}$ of the class S_W. Let f be an arbitrary element of $\mathfrak{N}$. Then f has a unique resolution $f = f^+ + f^-$, where f^+ is in $\mathfrak{M}^+$, f^- in $\mathfrak{M}^-$. Furthermore,

$$(f, Wf) = (f^+ + f^-, if^+ - if^-) = -i(f^+, f^+) + i(f^-, f^-) = 0.$$

Thus $|f^-| = |f^+|$, and we can set $f^- = -Vf^+$, where V is an operator with domain the set of elements of $\mathfrak{M}^+$ which are projections of elements of $\mathfrak{N}$ and is uniquely defined at every point of its domain. Since $\mathfrak{N}$ is a linear manifold and the projections with ranges $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are linear transformations, V is a linear transformation. Since, as we have already seen, V preserves norm, V is isometric. It is obvious that V belongs to the class $\mathcal{U}_W$ and that $\mathfrak{N}$ is the range of $I - V$. On the other hand, if V is an arbitrary element of the class $\mathcal{U}_W$, we have, for all f and g in the domain of V,

$$(f - Vf, W(g - Vg)) = (f - Vf, ig + iVg) = -i(f, g) + i(Vf, Vg) = 0.$$

Thus the range of $I - V$ belongs to S_W. Since it is evident that two isometric transformations V and V_1 with domains in $\mathfrak{M}^+$ and ranges in $\mathfrak{M}^-$ cannot satisfy the relation $\mathfrak{R}(I - V) = \mathfrak{R}(I - V_1)$ unless $V \equiv V_1$, the first assertion of the theorem is established.

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† We use the ordinary plus sign here and throughout this paper to indicate the *linear* sum of manifolds.

We now turn to the second. If V is an arbitrary member of the class $\mathcal{U}_W$ and $\{f_n\}$ an arbitrary convergent sequence in the domain of V, the sequence $\{f_n - Vf_n\}$ also converges, evidently to an element $f - Vf$ of $\overline{\mathfrak{N}}$. Furthermore, if $\{f_n\}$ is an arbitrary convergent sequence in $\mathfrak{N}$ and $f_n = f_n^+ - Vf_n^+$, the sequences $\{f_n^+\}$ and $\{Vf_n^+\}$ must converge separately since they belong to orthogonal linear manifolds. But then the first converges to an element f^+ in the domain of $\tilde{V}$, the second to $\tilde{V}f^+$. Hence if $\mathfrak{N}$ and V are in correspondence, so also are $\overline{\mathfrak{N}}$ and $\tilde{V}$. Since the correspondence is one-to-one, it follows that $\mathfrak{N}$ is closed if and only if V is closed.

Purely on the basis of the definition of the correspondence between S_W and $\mathcal{U}_W$ it is readily verified that $\mathfrak{N}_1 \subset \mathfrak{N}_2$ implies $V_1 \subseteq V_2$ while $V_1 \subset V_2$ implies $\mathfrak{N}_1 \subseteq \mathfrak{N}_2$, where, in either case, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ correspond to V_1 and V_2, respectively. Since the correspondence is one-to-one, $V_1 \equiv V_2$ if and only if $\mathfrak{N}_1 = \mathfrak{N}_2$. Hence $\mathfrak{N}_1 \subset \mathfrak{N}_2$ implies $V_1 \subset V_2$, and conversely. Furthermore, since a maximal isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ has no proper maximal isometric extension from $\mathfrak{M}^+$ to $\mathfrak{M}^-$, the equivalence, when $\mathfrak{N}$ and V correspond, of the statements "$\mathfrak{N}$ is maximal W-symmetric" and "V is a maximal isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$" follows at once.

To establish the relation

$$\mathfrak{M} \ominus \mathfrak{N} = W\mathfrak{N} + (\mathfrak{M}^+ \ominus \mathfrak{D}(V)) + (\mathfrak{M}^- \ominus \mathfrak{R}(V)),$$

where $\mathfrak{N}$ is a closed linear manifold in S_W, V the corresponding member of $\mathcal{U}_W$, we first observe that it is equivalent to the equation

$$\mathfrak{M} \ominus [(\mathfrak{M}^+ \ominus \mathfrak{D}(V)) + (\mathfrak{M}^- \ominus \mathfrak{R}(V))] = \mathfrak{N} + W\mathfrak{N}.$$

But, since $\mathfrak{M}^- = \mathfrak{M} \ominus \mathfrak{M}^+$, we have

$$\mathfrak{M} \ominus [(\mathfrak{M}^+ \ominus \mathfrak{D}(V)) + (\mathfrak{M}^- \ominus \mathfrak{R}(V))] = \mathfrak{D}(V) + \mathfrak{R}(V).$$

Hence we need only show that

$$\mathfrak{N} + W\mathfrak{N} = \mathfrak{D}(V) + \mathfrak{R}(V).$$

As every element of $\mathfrak{N}$ can be written in the form $f^+ - Vf^+$ and every element of $W\mathfrak{N}$ in the form $g^+ + Vg^+$, where f^+ and g^+ belong to $\mathfrak{D}(V)$, we have immediately $\mathfrak{N} + W\mathfrak{N} \subseteq \mathfrak{D}(V) + \mathfrak{R}(V)$. On the other hand, if f^+ is an arbitrary element of $\mathfrak{D}(V)$, $(f^+ - Vf^+)/2$ is in $\mathfrak{N}$ while $(f^+ + Vf^+)/2$ is in $W\mathfrak{N}$, whence we conclude that

$$f^+ = (f^+ - Vf^+)/2 + (f^+ + Vf^+)/2$$

belongs to $\mathfrak{N} + W\mathfrak{N}$. Similarly, if Vg^+ is an arbitrary element of $\mathfrak{R}(V)$, $-(g^+ - Vg^+)/2$ and $(g^+ + Vg^+)/2$ belong to the respective manifolds $\mathfrak{N}$ and

$W\mathfrak{N}$. Hence Vg^+ is in $\mathfrak{N}+W\mathfrak{N}$. Thus $\mathfrak{N}+W\mathfrak{N}=\mathfrak{D}(V)+\mathfrak{N}(V)$ as we wished to show.

That $\mathfrak{N}$ is hypermaximal W-symmetric if and only if the corresponding member of $\mathcal{U}_W$ is a unitary transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ is an immediate consequence of the result established in the preceding paragraph. The proof of the theorem is therefore complete.

Throughout the remainder of this chapter, W, $\mathfrak{M}$, $\mathfrak{M}^+$, $\mathfrak{M}^-$, $\mathcal{U}_W$, S_W have the same meanings as in Theorem 2.2. We shall hereafter write $\mathfrak{N}=\mathfrak{N}(V)$ and $V=V(\mathfrak{N})$ to indicate that $\mathfrak{N}$ and V are corresponding members of S_W and $\mathcal{U}_W$, respectively.

THEOREM 2.3. *Let $\mathfrak{M}_1^+$, $\mathfrak{M}_2^+$, $\cdots$, $\mathfrak{M}_n^+$ be mutually orthogonal linear manifolds in $\mathfrak{M}^+$. Let V_1, V_2, $\cdots$, V_n be transformations of the class $\mathcal{U}_W$ with domains respectively $\mathfrak{M}_1^+$, $\mathfrak{M}_2^+$, $\cdots$, $\mathfrak{M}_n^+$, and ranges mutually orthogonal. Then the manifolds $\mathfrak{N}(V_1)$, $\mathfrak{N}(V_2)$, $\cdots$, $\mathfrak{N}(V_n)$ are mutually orthogonal. If V is the linear transformation with domain $\mathfrak{M}_1^+ + \mathfrak{M}_2^+ + \cdots + \mathfrak{M}_n^+$ which is equal on $\mathfrak{M}_k$ to V_k, $(k=1, 2, \cdots, n)$, V belongs to $\mathcal{U}_W$ and $\mathfrak{N}(V)$ is $\mathfrak{N}(V_1)+\mathfrak{N}(V_2)+ \cdots +\mathfrak{N}(V_n)$. Conversely, if $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\cdots$, $\mathfrak{N}_n$ are mutually orthogonal linear manifolds in S_W, the domains, and the ranges, of $V(\mathfrak{N}_1)$, $V(\mathfrak{N}_2)$, $\cdots$, $V(\mathfrak{N}_n)$ are mutually orthogonal. If $\mathfrak{N}=\mathfrak{N}_1+\mathfrak{N}_2+ \cdots +\mathfrak{N}_n$, $\mathfrak{N}$ belongs to S_W and $V(\mathfrak{N})$ has domain the linear manifold determined by the domains of $V(\mathfrak{N}_1)$, $V(\mathfrak{N}_2)$, $\cdots$, $V(\mathfrak{N}_n)$ and is equal to $V(\mathfrak{N}_k)$ on the domain of the latter $(k=1, 2, \cdots, n)$.*

The truth of Theorem 2.3 follows from the definition of the correspondence of Theorem 2.2; we omit the details of proof.

DEFINITION 2.3. *The $(\mathfrak{M}^+, \mathfrak{M}^-)$-deficiency index of an isometric transformation V from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ is also called the W-deficiency index of V. If $\mathfrak{N}$ is a linear W-symmetric manifold and $V(\mathfrak{N})$ has the W-deficiency index (m, n), (m, n) is also said to be the W-deficiency index of $\mathfrak{N}$.*

THEOREM 2.4. *A necessary and sufficient condition that a closed linear manifold $\mathfrak{N}$ of the class S_W be maximal W-symmetric is that its W-deficiency index be either $(m, 0)$ or $(0, n)$. A necessary and sufficient condition that $\mathfrak{N}$ be hypermaximal W-symmetric is that its W-deficiency index be $(0, 0)$.*

The preceding theorem is merely a restatement, in terms of the terminology introduced in Definition 2.3 of the last two assertions of Theorem 2.2.

THEOREM 2.5. *Let j be the dimension number of $\mathfrak{M}^+$, k the dimension number of $\mathfrak{M}^-$. Then, if either j or k is zero, S_W contains only the manifold $\mathfrak{D}$ in $\mathfrak{M}$. If neither j nor k is zero, the class of all maximal W-symmetric manifolds in S_W has the cardinal number of the continuum.*

If $j = k < \aleph_0$, every maximal W-symmetric manifold in the class $\mathfrak{S}_W$ is hyper-maximal; that is, has the W-deficiency index $(0, 0)$. If $j > k$ $(k > j)$ holds, every maximal W-symmetric manifold in $\mathfrak{S}_W$ has the W-deficiency index $(j - k, 0)$ $((0, k - j))$. If $j = k = \aleph_0$ and p is a cardinal number on the range $0 \leq p \leq \aleph_0$, the class of all maximal W-symmetric manifolds with the W-deficiency index $(p, 0)$ $((0, p))$ has the cardinal number of the continuum.

If $j = k = \aleph_0$, or if $j > k$ $(k > j)$ holds, and $\mathfrak{M}_1$ is an arbitrary closed linear manifold in $\mathfrak{M}^+$ $(\mathfrak{M}^-)$ such that $\mathfrak{M}^+ \ominus \mathfrak{M}_1$ $(\mathfrak{M}^- \ominus \mathfrak{M}_1)$ has the dimension number $k(j)$, the class of all isometric transformations of $\mathfrak{M}^+ \ominus \mathfrak{M}_1$ into $\mathfrak{M}^-$ $(\mathfrak{M}^+$ into $\mathfrak{M}^- \ominus \mathfrak{M}_1)$ is in one-to-one correspondence according to Theorem 2.2 with a subclass of the class of all maximal W-symmetric manifolds in $\mathfrak{S}_W$. Each member of the subclass of $\mathfrak{S}_W$ so defined has the W-deficiency index $(p, 0)$ $((0, p))$, where p is the dimension number of $\mathfrak{M}_1$.

Theorem 2.5 is for the most part a straightforward interpretation in terms of the class $\mathfrak{S}_W$ of the analysis of the maximal transformations in the class $\mathcal{U}_W$ which is provided by Theorem 3.1. We leave the details of verification to the reader.

While it is evidently possible to elaborate extensively the preceding analysis and, in particular, to consider the problem of determining all of the maximal W-symmetric extensions of a given W-symmetric manifold, we refrain from such elaboration as being both unnecessary for the applications which we intend to make and also clearly suggested in outline by the analysis which we have already developed and by works of similar nature to which we referred at the beginning of this chapter.

3. **Illustrations and applications.** We pass instead to the consideration of two special types of transformation W which are of considerable importance and interest.

THEOREM 2.6. *Let $\mathfrak{M} = \mathfrak{H} \oplus \mathfrak{H}$ where $\mathfrak{H}$ is a unitary or Hilbert space, and let W be the transformation which takes every vector $\{f_1, f_2\}$ of $\mathfrak{H}$ into $i\{f_1, -f_2\}$. Then W is a unitary transformation in $\mathfrak{M}$ and $W^2 + I \equiv 0$. The characteristic manifolds $\mathfrak{M}^+$ and $\mathfrak{M}^-$ of W for the characteristic values $+i$ and $-i$ are $\mathfrak{H} + \mathfrak{O}$ and $\mathfrak{O} + \mathfrak{H}$, respectively. The class $\mathcal{U}_W$ is in one-to-one correspondence with the class of all isometric transformations X in $\mathfrak{H}$; V in $\mathcal{U}_W$ corresponds to X if and only if $\mathfrak{D}(V) = \mathfrak{D}(X) + \mathfrak{O}$ and $V\{f, 0\} = \{0, Xf\}$ for all f in the domain of X. V is a maximal isometric (unitary) transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ if and only if X is maximal isometric (unitary) in $\mathfrak{H}$. If V and X correspond, $\mathfrak{N}(V)$ is the graph of $-X$.*

The statements regarding W are evident. The remaining assertions of the

theorem are all proved by simple recourse to the definitions of the various terms and symbols involved.

THEOREM 2.7. *Let* $\mathfrak{M} = \mathfrak{H} \oplus \mathfrak{H}$ *where* $\mathfrak{H}$ *is a unitary or Hilbert space. Let* W *be the transformation with domain* $\mathfrak{M}$ *which takes* $\{f, g\}$ *into* $\{g, -f\}$. *Then* W *is unitary and* $W^2 + I \equiv O$. *The manifolds* $\mathfrak{M}^+$ *and* $\mathfrak{M}^-$ *in* $\mathfrak{M}$ *are, respectively, the set of vectors of the form* $\{h, ih\}$ *and the set of vectors of the form* $\{h, -ih\}$. *There is a one-to-one correspondence between the class* $\mathcal{U}_W$ *and the class* $\mathcal{X}$ *of all isometric transformations* X *in* $\mathfrak{H}$; V *and* X *correspond provided* $\{h, ih\}$ *is in the domain of* V *when and only when* h *is in the domain of* X *and* $V\{h, ih\} = \{Xh, -iXh\}$.

The class $\mathcal{S}_W$ *contains a subclass of manifolds each of which is the graph* $\mathfrak{B}(H)$ *of a symmetric transformation* H *in* $\mathfrak{H}$; *if* H *is an arbitrary linear symmetric transformation in* $\mathfrak{H}$, $\mathfrak{B}(H)$ *belongs to* $\mathcal{S}_W$. *If* $\mathfrak{B}(H) = \mathfrak{N}(V)$ *and* V *corresponds to the member of* X *of* $\mathcal{X}$ *according to the laws stated above,* $H \equiv i(I+X)(I-X)^{-1}$. H *is maximal symmetric (self-adjoint) if and only if* $\mathfrak{B}(H)$ *is maximal (hypermaximal)* W-*symmetric.*

Again the statements concerning W are evident from inspection. In verification of the correspondence between $\mathcal{U}_W$ and $\mathcal{X}$, we observe first that every isometric transformation X in $\mathcal{X}$ clearly generates a member V of the class $\mathcal{U}_W$ according to the following rules:

(1) $\mathfrak{D}(V)$ contains $\{h, ih\}$ if and only if the domain of X contains h;

(2) $V\{h, ih\} = \{Xh, -iXh\}$.

On the other hand, if ˘ belongs to $\mathcal{U}_W$, the equality $2|h|^2 = 2|k|^2$ which is valid whenever $\{k, -ik\} = V\{h, ih\}$ and the linear character of V imply that V generates a member X of $\mathcal{X}$ such that (1) and (2) are satisfied.

If H is a linear symmetric transformation in $\mathfrak{H}$, then $W\mathfrak{B}(H)$ is the set of vectors $\{Hf, -f\}$ in $\mathfrak{H} \oplus \mathfrak{H}$; hence the symmetry of H implies that $\mathfrak{B}(H)$ is in $\mathcal{S}_W$. If $\mathfrak{B}(H) = \mathfrak{N}(V)$ and V corresponds to X in $\mathcal{X}$, the domain of H is the range of $I-X$ and $H(I-X) \equiv i(I+X)$. The fact that $I-X$ has range dense in $\mathfrak{H}$ is readily shown to imply that it has no zeros and we thus have $H \equiv i(I+X)(I-X)^{-1}$. The statements "$H$ is maximal symmetric" and "$\mathfrak{B}(H)$ is maximal W-symmetric" are obviously equivalent by definition as also are the statements "H is self-adjoint" and "$\mathfrak{B}(H)$ is hypermaximal W-symmetric."

Further application of the theory of W-symmetric manifolds which we have developed, and of the more detailed theory which we have suggested to the study of the graphs of symmetric transformations would clearly yield new proofs of many of the known facts in the familiar theory of the connection between isometric and symmetric transformations. We prefer, however,

to postpone consideration of this theory to a later point where we are led to it again from a different direction. Here we shall prove only a variation of one known theorem which is intimately connected with the previously stated fact that a reduction operator A can be defined on the graph of the adjoint of any symmetric transformation.

THEOREM 2.8. *Let H be a closed linear symmetric transformation in Hilbert space $\mathfrak{H}$, and let $\mathfrak{D}^+$ and $\mathfrak{D}^-$ be the characteristic manifolds of H^* for the characteristic values $+i$ and $-i$, respectively. Then $\mathfrak{B}(H^*)\ominus\mathfrak{B}(H)$ is the set of elements of $\mathfrak{B}(H^*)$ which can be written in the form $\{f^+ + f^-, if^+ - if^-\}$, where f^+ is in $\mathfrak{D}^+$, f^- in $\mathfrak{D}^-$.*†

Let W, $\mathfrak{M}^+$, and $\mathfrak{M}^-$ have meanings the same as in Theorem 2.7, and let V be the isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ corresponding to $\mathfrak{B}(H)$ in accordance with that theorem. Since H is closed and linear, $\mathfrak{B}(H)$ is a closed linear manifold, and V is closed also. Moreover, according to equation (2.1), $(\mathfrak{H}\oplus\mathfrak{H})\ominus\mathfrak{B}(H)$ is the manifold

$$W\mathfrak{B}(H) \oplus (\mathfrak{M}^+ \ominus \mathfrak{D}(V)) \oplus (\mathfrak{M}^- \ominus \mathfrak{R}(V)).$$

But, by definition,

$$(\mathfrak{H} \oplus \mathfrak{H}) \ominus \mathfrak{B}(H) = W\mathfrak{B}(H^*).$$

Thus we have

$$W\mathfrak{B}(H) \oplus (\mathfrak{M}^+ \ominus \mathfrak{D}(V)) \oplus (\mathfrak{M}^- \ominus \mathfrak{R}(V)) = W\mathfrak{B}(H^*),$$

and, since the three components on the left are mutually orthogonal, this is equivalent to the equation

$$W\mathfrak{B}(H^*) \ominus W\mathfrak{B}(H) = (\mathfrak{M}^+ \ominus \mathfrak{D}(V)) \oplus (\mathfrak{M}^- \ominus \mathfrak{R}(V)).$$

But W is unitary, equal to iI on $\mathfrak{M}^+$, and to $-iI$ on $\mathfrak{M}^-$, so that the latter equation is equivalent to

$$\mathfrak{B}(H^*) \ominus \mathfrak{B}(H) = (\mathfrak{M}^+ \ominus \mathfrak{D}(V)) \oplus (\mathfrak{M}^- \ominus \mathfrak{R}(V)).$$

Hence to complete the proof it is necessary only to show that $(\mathfrak{M}^+\ominus\mathfrak{D}(V))$ consists of all vectors $\{f^+, if^+\}$ such that f^+ is in $\mathfrak{D}^+$ and $(\mathfrak{M}^-\ominus\mathfrak{D}(V))$ of all vectors $\{f^-, if^-\}$ such that f^- is in $\mathfrak{D}^-$.

Consider first an arbitrary element f^+ of $\mathfrak{D}^+$. Then $H^*f^+ = if^+$, and

$$(g, f^+) + (Hg, if^+) = i(g, f^+) - i(Hg, f^+) = 0$$

for all g in $\mathfrak{D}(H)$; and $\{f^+, if^+\}$ is perpendicular to $\mathfrak{B}(H)$. But $\mathfrak{B}(H) = \mathfrak{R}(I - V)$ and $\mathfrak{R}(V)$ is in $\mathfrak{M}^- = (\mathfrak{H}\oplus\mathfrak{H})\ominus\mathfrak{M}^+$. Therefore, since $(f^+, h) + (if^+, -ih) = 0$ for

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† Compare Stone, Theorem 9.4, and Teichmüller, loc. cit., pp. 104–105.

all h in $\mathfrak{H}$, $\{f^+, if^+\}$ is in $\mathfrak{M}^+\ominus\mathfrak{D}(V)$. On the other hand, if $\{f^+, if^+\}$ is in $\mathfrak{M}^+\ominus\mathfrak{D}(V)$, it is perpendicular to $\mathfrak{B}(H)$ and we have

$$(g, if^+) - (Hg, f^+) = -i(g, f^+) + (Hg, if^+) = 0$$

for all g in $\mathfrak{D}(H)$, whence it follows that $if^+ = H^*f^+$.

A similar argument establishes the analogous relation between $\mathfrak{M}^-\ominus\mathfrak{R}(V)$ and $\mathfrak{D}^-$, and completes the proof of the theorem.

THEOREM 2.9. *Let A be the projection in $\mathfrak{B}(H^*)$ with range $\mathfrak{B}_1(H^*)$ $=\mathfrak{B}(H^*)\ominus\mathfrak{B}(H)$. Then A is a reduction operator for H^*, with range-space $\mathfrak{B}_1(H^*)$. The transformation W associated with A by Definition 1.1 is equal on its domain to the operator in $\mathfrak{H}\oplus\mathfrak{H}$ which takes $\{f, g\}$ into $-\{g, -f\}$, where f and g are both in $\mathfrak{H}$.*

It is necessary for the proof of the theorem only to determine

$$((\mathfrak{H}\oplus\mathfrak{H})\oplus\mathfrak{B}_1(H^*))\ominus\mathfrak{B}(A).$$

Letting W_1 have the same meaning as W in Theorem 2.7, we have $\mathfrak{H}\oplus\mathfrak{H}$ $=\mathfrak{B}(H^*)\oplus W_1\mathfrak{B}(H)$. Hence

$$((\mathfrak{H}\oplus\mathfrak{H})\oplus\mathfrak{B}_1(H^*))\ominus\mathfrak{B}(A) = ((\mathfrak{B}(H^*)\oplus W_1\mathfrak{B}(H))\oplus\mathfrak{B}_1(H^*))\ominus\mathfrak{B}(A),$$

and, since $W_1\mathfrak{B}(H)+\mathfrak{D}$ is obviously orthogonal to $\mathfrak{B}(A)$ in $(\mathfrak{H}\oplus\mathfrak{H})\oplus\mathfrak{B}_1(H^*)$, we have now to determine

$$(\mathfrak{B}(H^*)\oplus\mathfrak{B}_1(H^*))\ominus\mathfrak{B}(A).$$

But, considered as a transformation from $\mathfrak{B}(H^*)$ to $\mathfrak{B}_1(H^*)$, A has an adjoint from $\mathfrak{B}_1(H^*)$ to $\mathfrak{B}(H^*)$, which is clearly equal on its domain to the identity. Therefore, in accordance with Theorem 2.8,

$$((\mathfrak{H}\oplus\mathfrak{H})\oplus\mathfrak{B}_1(H^*))\ominus\mathfrak{B}(A)$$

consists of those and only those elements of $(\mathfrak{H}\oplus\mathfrak{H})\oplus\mathfrak{B}_1(H^*)$ which are of the form

$$\{Hf + if^+ - if^-, -f - f^+ - f^-, \{-if^+ + if^-, f^+ + f^-\}\}.$$

Since, when we again take account of Theorem 2.8, these elements are revealed as precisely those elements of $(\mathfrak{H}\oplus\mathfrak{H})\oplus\mathfrak{B}_1(H^*)$ which are of the form $\{H^*g, -g, WAg\}$, where W has the character stated in the theorem, the proof is complete.

Theorem 2.6 gives complete information about the structure of the manifolds $\mathfrak{N}$ of the class S_W for the special operator W there considered. Theorem 2.7, however, provides this information only in the special case that $\mathfrak{N}$ is the graph of a symmetric transformation. It is desirable, therefore, that we dis-

cuss more fully the general case arising under Theorem 2.7. An analysis adequate for our purposes is provided by the theorem which we prove next. Following von Neumann, we shall call a transformation H in a space $\mathfrak{H}$ Hermitian if the equation $(f, Hg) - (Hf, g) = 0$ is satisfied for all f and g in the domain of H.† Thus a Hermitian transformation is symmetric if its domain determines $\mathfrak{H}$. We emphasize that $\mathfrak{H}$ may have a finite dimension number and also that we do not admit, as von Neumann does in the work just referred to, many-valued operators H.

THEOREM 2.10. *Let $\mathfrak{M}$ and W have the same meanings as in Theorem 2.7, and let $\mathfrak{N}$ be an arbitrary closed linear manifold in the class S_W. Let X be the member of the class X corresponding to $V(\mathfrak{N})$ in accordance with Theorem 2.7. Then X is a closed transformation. Let $\mathfrak{D}_1(X)$ be the characteristic manifold of X for the characteristic value 1, and let $\mathfrak{D}_0(X) = \mathfrak{D}(X) \ominus \mathfrak{D}_1(X)$. Let X_0 be the contraction of X with domain $\mathfrak{D}_0(X)$, and let $\mathfrak{H}_0$ be the closed linear manifold determined by the linear sum of the domain and range of X_0. Then $\mathfrak{H}_0$ is in $\mathfrak{H} \ominus \mathfrak{D}_1(X)$, $(I - X_0)^{-1}$ exists, and the transformation*

$$H \equiv i(I + X_0)(I - X_0)^{-1}$$

is a closed linear Hermitian transformation in $\mathfrak{H}_0$. If $\{f, g\}$ is an arbitrary element of $\mathfrak{N}$, f is in the domain of H and $g = Hf + h$, where h belongs to $\mathfrak{D}_1(X)$. The resolution of $\{f, g\}$ so provided is unique. $\mathfrak{N}$ is maximal (hypermaximal) W-symmetric if and only if $\mathfrak{H}_0 = \mathfrak{H} \ominus \mathfrak{D}_1(X)$ and H is maximal symmetric (self-adjoint) in $\mathfrak{H}_0$.

That X is closed follows at once from the fact that $\mathfrak{N}$ is closed, when we take account of the relation between X and $V(\mathfrak{N})$ described in Theorem 2.7.

By definition, $\mathfrak{H}_0 = \mathfrak{D}(X_0) \oplus \mathfrak{R}(X_0)$. Since, also by definition, $\mathfrak{D}(X_0) = \mathfrak{D}(X) \ominus \mathfrak{D}_1(X)$, it follows that $\mathfrak{D}(X_0) \subseteqq \mathfrak{H} \ominus \mathfrak{D}_1(X)$. But $X_0 \subseteqq X$ and X is isometric. Therefore, as X is defined throughout $\mathfrak{D}_1(X)$ and leaves it invariant, we must have $\mathfrak{R}(X_0) \subseteqq \mathfrak{H} \ominus \mathfrak{D}_1(X)$. Consequently $\mathfrak{H}_0 \subseteqq \mathfrak{H} \ominus \mathfrak{D}_1(X)$.

Since $(I - X_0)f = 0$ implies $(I - X)f = 0$ which implies in turn that f belongs to $\mathfrak{D}_1(X)$, it follows that $(I - X_0)^{-1}$ exists. That the transformation H of the theorem is a closed linear Hermitian transformation in the space $\mathfrak{H}_0$ is readily verified on the basis of its definition in terms of X_0.

If $\{f, g\}$ is an arbitrary element of $\mathfrak{N}$, then, according to Theorem 2.7, $f = k - Xk$, $g = i(k + Xk)$, where k is an element of $\mathfrak{D}(X)$. Let $k = k_1 + k_2$, where k_1 belongs to $\mathfrak{D}_0(X)$, k_2 to $\mathfrak{D}_1(X)$. Then $f = k_1 - Xk_1$, $g = i(k_1 + X_0 k_1) + 2i k_2$. Thus $g = Hf + h$, where $h = 2i k_2$. That this resolution of $\{f, g\}$ is unique is an

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immediate consequence of the uniqueness of the resolutions $k = k_1 + k_2$ for k, and

$$\{f, g\} = \{k - Xk, i(k + Xk)\}$$

for $\{f, g\}$.

Now let us suppose that $\mathfrak{N}$ is maximal W-symmetric. Then, by Theorem 2.7, either the domain or the range of X is identically $\mathfrak{H}$. Consequently, either the domain or the range of X_0 is identically $\mathfrak{H}_0$ and $\mathfrak{H}_0 = \mathfrak{H} \ominus \mathfrak{D}_1(X)$. Furthermore, the range of $I - X_0$ is dense in $\mathfrak{H}_0$. For let the equation $(f, g - X_0 g) = 0$ be satisfied for all g in the domain of X_0 and some element f of $\mathfrak{H}_0$. Then $(f, g) - (f, X_0 g) = 0$. Hence, if X_0 has domain $\mathfrak{H}_0$, we have $f = X_0^* f$. But it is readily shown that X_0^* is equal to X_0^{-1} on $\mathfrak{R}(X_0)$ and equal to 0 on $\mathfrak{H} \ominus \mathfrak{R}(X_0)$. Thus f is in $\mathfrak{R}(X_0)$ and $f = X_0^{-1} f$. Since the inverse of $I - X_0$ exists, it follows that $f = 0$. Similarly, if X_0 has range $\mathfrak{H}_0$, we have $f = (X_0^{-1})^* f$ and a parallel argument leads again to the conclusion $f = 0$. Thus $H \equiv i(I + X_0)(I - X_0)^{-1}$ is symmetric in $\mathfrak{H}_0$ and, since either the domain or the range of X_0 is identically $\mathfrak{H}_0$, H is maximal as well. Furthermore, when $\mathfrak{N}$ is hypermaximal W-symmetric, X_0 has domain and range identically $\mathfrak{H}_0$ so that in this case H is self-adjoint.

Next let us suppose that $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{D}_1(X)$ and that H is maximal symmetric in $\mathfrak{H}_0$. Then $\mathfrak{N} = \mathfrak{B}(H) \oplus \mathfrak{P}$ where $\mathfrak{P}$ is the manifold in $\mathfrak{M} = \mathfrak{H} \oplus \mathfrak{H}$ whose elements are of the form $\{0, h\}$, h in $\mathfrak{D}_1(X)$. Thus $\mathfrak{M} \ominus \mathfrak{N} = W\mathfrak{B}(H^*) \oplus W\mathfrak{P}$, and in view of Theorem 2.8 this equation reveals immediately that $\mathfrak{N}$ is maximal W-symmetric. Furthermore, if $H \equiv H^*$, then $\mathfrak{B}(H^*) = \mathfrak{B}(H)$, and $\mathfrak{N}$ is clearly hypermaximal W-symmetric.

We bring this section to a close with a simple theorem which is revealed later as of fundamental importance.

THEOREM 2.11. *Let A be a reduction operator with domain in the graph of the adjoint H^* of a symmetric transformation H in a Hilbert space $\mathfrak{H}$. Let $\mathfrak{M}$ be the range-space of A, and let W be the unitary transformation associated with A by Definition 1.1. Let $\mathfrak{M}^+$ and $\mathfrak{M}^-$ be the characteristic manifolds of W for the characteristic values $+i$ and $-i$, respectively. Let U be the transformation in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ which takes $\{f_1, f_2, h\}$ into $\{f_2, -f_1, Wh\}$. Then U is a unitary transformation in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ and $U^2 + I \equiv O$. The graph $\mathfrak{B}(A)$ of A, consisting of all vectors of the form $\{f, H^*f, Af\}$ in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ is a hypermaximal U-symmetric manifold. The characteristic manifolds of U for the characteristic values $+i$ and $-i$ are, respectively, the manifold $\mathfrak{L}^+$ consisting of all elements of the form $\{f, if, h_+\}$, where f is in $\mathfrak{H}$, h_+ in $\mathfrak{M}^+$, and the manifold $\mathfrak{L}^-$ of all elements of the form $\{f, -if, h_-\}$, where f is in $\mathfrak{H}$ and h_- in $\mathfrak{M}^-$.*

Since it is an immediate consequence of Definition 1.1 that

$$U\mathfrak{B}(A) = (\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}) \ominus \mathfrak{B}(A),$$

it is necessary only to establish that U has the properties stated in the theorem. That $U^2 + I \equiv O$ is readily verified:

$$U^2\{f_1, f_2, h\} = U\{f_2, -f_1, Wh\} = \{-f_1, -f_2, W^2h\}$$

and $W^2h = -h$. From inspection it is evident that $\mathfrak{L}^+$ and $\mathfrak{L}^-$ are at least subsets of the indicated characteristic manifolds of U; that they must be identically those manifolds is an immediate consequence of the easily proved equality

$$\mathfrak{L}^+ + \mathfrak{L}^- = \mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}.$$

4. Real W-symmetric manifolds. Just as it is desirable to consider the special properties of symmetric transformations which are real with respect to a conjugation, so here it is desirable to consider "real" W-symmetric manifolds.

We recall that a conjugation J in a Hilbert space or unitary space $\mathfrak{M}$ is a one-to-one transformation of $\mathfrak{M}$ into itself such that

$$J^2 \equiv I, \quad (Jf, Jg) = \overline{(f, g)} = (g, f),$$

and that a transformation T in $\mathfrak{M}$ is said to be real with respect to J if it permutes with J.†

DEFINITION 2.4. *A linear manifold $\mathfrak{N}$ in a unitary or Hilbert space $\mathfrak{M}$ is said to be real with respect to a conjugation if it contains Jf whenever it contains f.*

THEOREM 2.12. *If a linear manifold $\mathfrak{N}$ in $\mathfrak{M}$ is real with respect to a conjugation J, then the closed linear manifold which $\mathfrak{N}$ determines is real with respect to J.*

For a linear manifold $\mathfrak{N}$ to be real with respect to J it is necessary and sufficient that (1) $J\mathfrak{N} = \mathfrak{N}$; *if $\mathfrak{N}$ is also closed, either of the following conditions is necessary and sufficient:* (2) $\mathfrak{M} \ominus \mathfrak{N}$ *is real with respect to J,* (3) $(\mathfrak{M} \ominus \mathfrak{N}) = J(\mathfrak{M} \ominus \mathfrak{N})$.

Since a conjugation J is evidently continuous, it follows at once that the closed linear manifold $\overline{\mathfrak{N}}$ determined by a linear manifold $\mathfrak{N}$ is real with respect to J, if $\mathfrak{N}$ itself is.

If $\mathfrak{N}$ is real with respect to J, then $J\mathfrak{N}$ is a subset of $\mathfrak{N}$; if it is a proper subset, then $J^2\mathfrak{N}$ is also a proper subset of $\mathfrak{N}$. The latter is clearly impossible, however, since $J^2 \equiv I$. Thus the condition $J\mathfrak{N} = \mathfrak{N}$ is necessary for the reality of $\mathfrak{N}$ with respect to J; that it is sufficient is obvious. For $\mathfrak{N}$ closed, the neces-

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† Stone, pp. 357–365.

sity and sufficiency of the condition (3) now follow from the fact that $(Jf, Jg) = \overline{(f, g)}$ for all f in $\mathfrak{N}$ and g in $\mathfrak{M} \ominus \mathfrak{N}$, and the fact that J takes $\mathfrak{N}$ into itself and (1). Since (2) is, by (1), equivalent to (3), the proof is complete.

LEMMA 2.1. *A unitary transformation* W, $W^2 + I \equiv O$, *permutes with a conjugation* J *if and only if* $\mathfrak{M}^- = J\mathfrak{M}^+$ *and* $\mathfrak{M}^+ = J\mathfrak{M}^-$.

The proof is immediate.

THEOREM 2.13. *Let* $\mathfrak{N}$ *be a closed* W-*symmetric manifold in* $\mathfrak{M}$, *and let* $V \equiv V(\mathfrak{N})$. *Then for* $\mathfrak{N}$ *to be real with respect to a conjugation* J *which permutes with* W, *it is necessary and sufficient that the identity* $V \equiv JV^{-1}J$ *hold.*

We observe first that $J\mathfrak{N}$ is W-symmetric, since the relations

$$W\mathfrak{N} \subseteq \mathfrak{M} \ominus \mathfrak{N}, \quad JW\mathfrak{N} \subseteq \mathfrak{M} \ominus J\mathfrak{N}, \quad WJ\mathfrak{N} \subseteq \mathfrak{M} \ominus J\mathfrak{N}$$

are equivalent. Also, if $\mathfrak{N} = \mathfrak{R}(I - V)$, then $J\mathfrak{N} = \mathfrak{R}(I - JV^{-1}J)$, and $JV^{-1}J$ is isometric with domain in $\mathfrak{M}^+$ and range in $\mathfrak{M}^-$ by Lemma 2.1. Hence, by Theorem 2.2, $\mathfrak{N} = J\mathfrak{N}$ if and only if $V \equiv JV^{-1}J$, and the proof is complete.

THEOREM 2.14. *Let* $\mathfrak{M} = \mathfrak{H} \oplus \mathfrak{H}$, *let* W *be as in Theorem 2.7, and let* J_0 *be a conjugation in* $\mathfrak{H}$. *Then the transformation* J *in* $\mathfrak{M}$ *defined by the equation* $J\{f, g\} = \{J_0 f, J_0 g\}$ *is a conjugation which permutes with* W. *A closed linear* W-*symmetric manifold* $\mathfrak{N}$ *in* $\mathfrak{M}$ *is real with respect to* J *if and only if the manifold* $\mathfrak{D}_1(X)$ *and the transformation* H *associated with* $\mathfrak{N}$ *by Theorem 2.10 are real with respect to* J_0.

The first assertion is a direct consequence of Lemma 2.1.

If H is real with respect to J_0, then $\mathfrak{B}(H)$ is real with respect to J, by definition. Furthermore, the manifold $\mathfrak{D} + \mathfrak{D}_1(X)$ in $\mathfrak{M} = \mathfrak{H} \oplus \mathfrak{H}$ is clearly real with respect to J if $\mathfrak{D}_1(X)$ is real with respect to J_0. Thus, since

$$\mathfrak{N} = \mathfrak{B}(H) + (\mathfrak{D} + \mathfrak{D}_1(X)),$$

$\mathfrak{N}$ is real with respect to J when H and $\mathfrak{D}_1(X)$ are real with respect to J_0.

On the other hand, suppose $\mathfrak{N}$ is real with respect to J. Then

$$J\mathfrak{B}(H) + (\mathfrak{D} + J_0\mathfrak{D}_1(X)) = \mathfrak{B}(H) + (\mathfrak{D} + \mathfrak{D}_1(X)).$$

But $J\mathfrak{B}(H) = \mathfrak{B}(J_0 H J_0)$ and $J_0\mathfrak{D}_1(X)$ is orthogonal to the range of $J_0 H J_0$. Furthermore $J_0 H J_0$ is readily shown to be a Hermitian transformation. But according to Theorem 2.10, H and $\mathfrak{D}_1(X)$ orthogonal to the range of H are uniquely determined by $\mathfrak{N}$; so we have

$$J_0 H J_0 \equiv H, \qquad J_0 \mathfrak{D}_1(X) = \mathfrak{D}_1(X).$$

Consequently H and $\mathfrak{D}_1(X)$ are real with respect to J_0.

CHAPTER III. REDUCTION OPERATORS

1. An alternative definition. In this chapter we examine in detail the situation described in Definition 1.1, with particular attention to the structure of transformations A satisfying the conditions of that definition.

Before proceeding, we introduce the following notations whose meanings remain fixed throughout the chapter:

DEFINITION 3.1. *H is a closed symmetric transformation in Hilbert space $\mathfrak{H}$, with domain $\mathfrak{D}$; $\mathfrak{D}^*$ is the domain of H^*; $\mathfrak{B}$ and $\mathfrak{B}^*$ are the graphs of H and H^*, respectively; $\mathfrak{D}^+$ and $\mathfrak{D}^-$ are the characteristic manifolds of H^* for the characteristic values $+i$ and $-i$, respectively; $\mathfrak{B}^+$ ($\mathfrak{B}^-$) is the subset of $\mathfrak{B}^*$ defined as follows: $\{f, H^*f\}$ in $\mathfrak{B}^*$ is in $\mathfrak{B}^+$ ($\mathfrak{B}^-$) if and only if f is in $\mathfrak{D}^+$ ($\mathfrak{D}^-$).*

We recall from Theorem 2.8 that $\mathfrak{B}^* \ominus \mathfrak{B} = \mathfrak{B}^+ + \mathfrak{B}^-$.

THEOREM 3.1. *If A is a reduction operator for H^*, the contraction A_1 of A with domain $[(\mathfrak{B}^+ + \mathfrak{B}^-) \cdot \mathfrak{D}(A)]$ is a closed linear transformation with domain dense in $\mathfrak{B}^+ + \mathfrak{B}^-$ and range dense in $\mathfrak{M}$, the range-space of A. The transformations A_1^* and A_1^{-1} exist and*

$$(3.1) \qquad\qquad A_1^* \equiv Q A_1^{-1} W,$$

where Q is the transformation in $\mathfrak{B}^+ + \mathfrak{B}^-$ which takes $\{f, g\}$ into $\{g, -f\}$, f and g being elements of $\mathfrak{H}$.

Conversely, if A_1 is any closed linear transformation with domain dense in $\mathfrak{B}^+ + \mathfrak{B}^-$ and range in a unitary or Hilbert space $\mathfrak{M}$, and if A_1^{-1} exists and satisfies (3.1) for some unitary transformation W in $\mathfrak{M}$, then the closed linear transformation A with domain $\mathfrak{D}(A_1) + \mathfrak{B}$ which is equal to O on $\mathfrak{B}$ and to A_1 on $\mathfrak{D}(A_1)$, is a reduction operator for H^*.

We note first that $\mathfrak{D}(A_1)$ is dense in $\mathfrak{B}^+ + \mathfrak{B}^-$, since $\mathfrak{B}$ is a closed linear manifold in $\mathfrak{D}(A)$ and $\mathfrak{D}(A)$ is dense in $\mathfrak{B}^*$. Thus the existence of A_1^* is assured. Furthermore, the existence of A_1^{-1} is assured by Theorem 1.1, and the relation $A_1^* \supseteq Q A_1^{-1} W$ follows at once from Definition 1.1.

Hence we have only to show that $Q A_1^{-1} W \supseteq A_1^*$; to do this we take direct recourse to the definition of the adjoint. Accordingly, we consider an arbitrary vector $\{f, H^*f, h\}$ in $(\mathfrak{B}^+ + \mathfrak{B}^-) \oplus \mathfrak{M}$ such that the equation

$$(g, f) + (H^*g, H^*f) - (Ag, H^*g, h) = 0$$

is satisfied for all $\{g, H^*g\}$ in $\mathfrak{D}(A_1)$. Since $\{f, H^*f\}$ is by hypothesis in $\mathfrak{B}^* \ominus \mathfrak{B}$, the equation is satisfied for all $\{g, H^*g\}$ in $\mathfrak{D}(A_1)$ if and only if it is satisfied for all $\{g, H^*g\}$ in $\mathfrak{D}(A)$. Thus, by Definition 1.1, we must have $\{f, H^*f\} = \{H^*k, -k\}$, $-h = WA_1\{k, H^*k\}$. Consequently, since $W \equiv -W^{-1}$,

$$\{k,\ H^*k\} = -A_{\bar{1}}^{-1}W^{-1}h = A_{\bar{1}}^{-1}Wh.$$

But $\{f,\ H^*f\} = A_1^*h$ and $\{f,\ H^*f\} = Q\{k,\ H^*k\}$. Therefore, whenever h is in the domain of A_1^*, Wh is in the domain of $A_{\bar{1}}^{-1}$ and $A_1^*h = QA_{\bar{1}}^{-1}Wh$. Thus $A_1^* \subseteq QA_{\bar{1}}^{-1}W$.

If, on the other hand, A_1 is given with the properties specified in the theorem, it is readily shown that its linear extension A defined by the relations $A = A_1$ on $\mathfrak{D}(A_1)$, $A = 0$ on $\mathfrak{B}$, satisfies the conditions of Definition 1.1 and so is a reduction operator for H^*.

Throughout the remainder of this paper, the subscript 1 attached to the symbol for a reduction operator indicates the contraction defined in Theorem 3.1. With this convention, we may take as the definition of a reduction operator the identity (3.1). The latter has the advantage of greater transparency; in particular, it reveals at once the following important information:

THEOREM 3.2. *The range of a reduction operator A for H^* is the entire range-space $\mathfrak{M}$ of A if and only if $\mathfrak{M}$ is a unitary space or A is bounded.*

If $\mathfrak{M}$ is unitary, A is necessarily bounded, so we may suppress consideration of the dimension number of $\mathfrak{M}$.

Obviously A is bounded if and only if A_1 is and thus if and only if A_1^* is. But A_1^* being closed is bounded if and only if its domain is $\mathfrak{M}$, and, by the identity (3.1), its domain is $\mathfrak{M}$ if and only if the domain of $A_{\bar{1}}^{-1}$ (that is, the range of A) is $\mathfrak{M}$.

On the basis of Theorem 3.2, it would be possible to proceed at once to the solution of the problem proposed at the end of Chapter I for the case that A is bounded. For unbounded reduction operators, however, more elaborate analysis is necessary.

2. **Characterization of all reduction operators.** We give now a characterization of all reduction operators which, except for minor points of detail, is due to M. H. Stone.

THEOREM 3.3. *If A is a reduction operator for H^*, with range-space $\mathfrak{M}$, and X is an arbitrary isometric transformation with domain $\mathfrak{M}$ and range $\mathfrak{N}$, then $C \equiv XA$ is a reduction operator for H^*, with range-space $\mathfrak{N}$. The unitary transformation in $\mathfrak{N}$ associated with C by Definition 1.1 is $U \equiv XWX^{-1}$. The characteristic manifolds of U for the characteristic values $+i$ and $-i$ are $X\mathfrak{M}^+$ and $X\mathfrak{M}^-$, respectively, where $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are the characteristic manifolds of W for the characteristic values $+i$ and $-i$, respectively.*

We set $C_1 \equiv XA_1$, obtaining $A_1 \equiv X^{-1}C_1$, $A_1^* \equiv C_1^*X$, $A_{\bar{1}}^{-1} \equiv C_{\bar{1}}^{-1}X$. Thus from the identity (3.1) we have $C_1^*X \equiv QC_{\bar{1}}^{-1}XW$, or $C_1^* \equiv QC_{\bar{1}}^{-1}XWX^{-1}$, which by Theorem 3.1 implies that $C \equiv XA$ is a reduction operator for H^*.

To prove the final assertions of the theorem, we make use of the identities $UX \equiv XW$, $X^{-1}U \equiv WX^{-1}$. If $Wh = ah$, we have, by the first identity, $UXh = Xah = aXh$. On the other hand, if $Uh = ah$, we have, by the second identity, $aX^{-1}h \equiv X^{-1}ah \equiv WX^{-1}h$. Setting a equal first to i and then to $-i$, we obtain at once the desired result.

DEFINITION 3.2. *If A and C are reduction operators for H^*, with range-spaces $\mathfrak{M}$ and $\mathfrak{N}$, respectively, C is said to be equivalent to A, $C \sim A$, if there exists an isometric transformation X with domain $\mathfrak{M}$ and range $\mathfrak{N}$ such that $C \equiv XA$.*

THEOREM 3.4. *Let A, C, and D be reduction operators for H^*. Then $C \sim A$ implies $A \sim C$, and $D \sim C$, $C \sim A$ imply $D \sim A$.*

The proof is immediate.

DEFINITION 3.3. *The class of all reduction operators for H^* which are equivalent to a single reduction operator A is called the equivalence class of A, or merely an equivalence class.*

We can now characterize all reduction operators for H^* by characterizing a representative member of each equivalence class. Our method is to show that $(C_1^* C_1)^{1/2} \equiv (A_1^* A_1)^{1/2}$ whenever A and C are equivalent and that $B_1 \equiv (A_1^* A_1)^{1/2}$ defines, in accordance with the second paragraph of Theorem 3.1, a reduction operator B for H^*, which is equivalent to A. Thus B, or B_1, completely determines the equivalence class of A. Our characterization of B_1 is based on the following theorem:

THEOREM 3.5. *Let $\mathfrak{L}$ be a Hilbert or unitary space. Let T be a nonnegative definite self-adjoint transformation in $\mathfrak{L}$ such that T^{-1} exists and satisfies the identity $T \equiv QT^{-1}R$, where Q and R are unitary transformations in $\mathfrak{L}$ such that $Q^2 \equiv R^2 \equiv -I$. Then $R \equiv Q^{-1}$, and the resolution of the identity $E(\lambda)$ in $\mathfrak{L}$ associated with T has the following properties:*
(1) $E(\lambda) \equiv O$ *for* $\lambda \leq 0$;
(2) $E(\lambda) \equiv Q(I - E(1/\lambda - 0))Q^{-1}$ *for* $0 < \lambda < \infty$;
(3) *the range $\mathfrak{N}$ of $E(1 - 0)$ is a Q-symmetric manifold;*
(4) *the range of $I - E(1)$ is $Q\mathfrak{N}$;*
(5) *the range $\mathfrak{P}$ of $E(1) - E(1 - 0)$ is of the form $\mathfrak{P} = \mathfrak{N}_1 + \mathfrak{N}_2$, where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are closed linear manifolds belonging to the characteristic manifolds of Q for the characteristic values i and $-i$, respectively.*

Let $\mathfrak{N}$ be an arbitrary closed Q-symmetric manifold in $\mathfrak{L}$, and let $E_{\mathfrak{N}}(\lambda)$ be a resolution of the identity in $\mathfrak{N}$ such that $E_{\mathfrak{N}}(\lambda) \equiv O$, $\lambda \leq 0$; $E_{\mathfrak{N}}(1 - 0) \equiv I$. Then the conditions

(6) $E(\lambda) = E_{\mathfrak{N}}(\lambda)E_{\mathfrak{N}}$,   *for* $\lambda < 1$;

(7) $E(1) = E_{\mathfrak{u}}$,

where $\mathfrak{U} = \mathfrak{L} \ominus Q\mathfrak{N}$;

$$(8) \qquad\qquad E(\lambda) = Q(I - E(1/\lambda - 0))Q^{-1}, \qquad for \quad 1 < \lambda < \infty,$$

define a resolution of the identity $E(\lambda)$ *in* $\mathfrak{L}$ *whose associated self-adjoint transformation* T *is nonnegative definite and has an inverse such that* $T \equiv QT^{-1}Q^{-1}$.

Since $T \equiv QT^{-1}R$ is self-adjoint, we have also $T \equiv R^{-1}T^{-1}Q^{-1}$, the identity $(QT^{-1}R)^* \equiv R^{-1}T^{-1}Q^{-1}$ being easily established. Hence $T^2 \equiv QT^{-2}Q^{-1}$, or $T^2 \equiv (QT^{-1}Q^{-1})^2$. As $QT^{-1}Q^{-1}$ and T are both nonnegative definite, we conclude that they are equal and thus that $R = Q^{-1}$ on $\mathfrak{D}(T)$. Since the latter manifold is dense in $\mathfrak{L}$, and R and Q^{-1} are both bounded, we have $R \equiv Q^{-1}$.

Now let $E(\lambda)$ be the resolution of the identity associated with T. Then the fact that T is nonnegative definite and has an inverse has as a consequence (1) of the theorem. To prove (2) we first recall that the unitary equivalence of two self-adjoint transformations is a necessary and sufficient condition for the unitary equivalence of the corresponding resolutions of the identity for each value of the real parameter λ which appears in the definition of both.† Since the resolution of the identity $F(\lambda)$ for T^{-1} is obtained from that for T according to the equations

$$F(\lambda) \equiv 0, \qquad\qquad\qquad\qquad\qquad \lambda \leqq 0,$$

$$F(\lambda) \equiv I - E(1/\lambda - 0), \qquad\qquad\qquad 0 < \lambda < \infty,$$

the unitary equivalence of T and T^{-1} under the unitary transformation Q yields the relation (2). From (2) we obtain at once $E(1-0) \equiv Q(I-E(1))Q^{-1}$ which implies (4); and, since the range of $I-E(1)$ is orthogonal to the range of $E(1-0)$, (3) now follows. In order to establish (5) we have only to observe that $E(1) - E(1-0)$ has range $\mathfrak{L} \ominus (\mathfrak{N} + Q\mathfrak{N})$ and then apply Theorem 2.2.

To prove the second part of the theorem we note first that the self-adjoint transformation T associated with the resolution of the identity defined by the relations (6), (7), and (8) has an inverse whose resolution of the identity $F(\lambda)$ is obtained from $E(\lambda)$ in the manner indicated above. Thus we have $E(\lambda) \equiv QF(\lambda)Q^{-1}$ which, according to a theorem already indicated, implies $T \equiv QT^{-1}Q^{-1}$.

THEOREM 3.6. *Each equivalence class* $\mathcal{A}$ *of reduction operators for* H^* *contains one and only one operator* B *with range-space* $\mathfrak{B}^+ + \mathfrak{B}^-$ *such that* B_1 *is a nonnegative definite self-adjoint transformation in* $\mathfrak{B}^+ + \mathfrak{B}^-$. *If* B *is such a reduction operator,* $B_1 \equiv QB_1^{-1}Q^{-1}$. *If* A *is an arbitrary reduction operator in the equivalence class of* B, *then* $A^*A \equiv B^2$ *and* $A_1^*A_1 \equiv B_1^2$.

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† Stone, Theorem 7.1.

Let the class $\mathcal{A}$ be given, and let C belong to $\mathcal{A}$. Then the transformation $C_1^*C_1$ is a nonnegative definite self-adjoint transformation and has a unique nonnegative definite square root B_1, $B_1^2 \equiv C_1^*C_1$, such that $B_1 \equiv XC_1$, where X is an isometric transformation with domain the closure of the range of C_1 and range the closure of the range of C_1^*.† Thus X is an isometric transformation with domain the range-space $\mathfrak{M}$ of C and range $\mathfrak{B}^+ + \mathfrak{B}^-$. By Theorem 3.3, $B \equiv XC$ belongs to $\mathcal{A}$. Since $B_1 \equiv QB_1^{-1}XWX^{-1}$, where W is the unitary operator in $\mathfrak{M}$ associated with C, we have by Theorem 3.5, $B_1 \equiv QB_1^{-1}Q^{-1}$. Now let A be any other member of the class $\mathcal{A}$. Then $A_1 \equiv YC_1$ and $A_1^* \equiv C_1^*Y^{-1}$, where Y is isometric. Thus $A_1^*A_1 \equiv B_1^2$, from which the equation $A^*A \equiv B^2$ follows at once, since $A_1^* \equiv A^*$. In particular, if A_1 is self-adjoint in $\mathfrak{B}^+ + \mathfrak{B}^-$, we have $A_1^2 \equiv B_1^2$. Consequently, if A_1 is also nonnegative definite, we must have $A_1 \equiv B_1$. Thus B is unique, and the proof is complete.

DEFINITION 3.4. *If A is a reduction operator for H^*, $E_A(\lambda)$ denotes the resolution of the identity in $\mathfrak{B}^+ + \mathfrak{B}^-$ associated with $(A_1^*A_1)^{1/2}$.*

Theorems 3.3, 3.5, and 3.6 evidently provide a constructive characterization of all reduction operators for H^*. This characterization, since it allows us to study an arbitrary reduction operator A in terms of the self-adjoint transformation $B_1 \equiv (A_1^*A_1)^{1/2}$ whose properties are described in Theorem 3.5, leads to comparatively simple proofs of many theorems which are otherwise established only with considerable difficulty. In particular, it reveals the effect of the deficiency index of H on the reduction operators for H^*.

THEOREM 3.7. *Let A be a reduction operator for H^*, and let $\mathfrak{M}$ and W have the same meanings as in Definition 1.1, $\mathfrak{M}^+$ and $\mathfrak{M}^-$ the same meanings as in Theorem 2.2. Let m and n be the dimension numbers of $\mathfrak{M}^+$ and $\mathfrak{M}^-$, respectively. Then (n, m) is the deficiency index of H.*

A necessary and sufficient condition that there be unbounded reduction operators for H^* is that H have the deficiency index $(\aleph_0, \aleph_0)$.

Evidently $\mathfrak{B}^-$ and $\mathfrak{B}^+$ are the characteristic manifolds of Q^{-1} for the characteristic values i and $-i$, respectively. But then, by Theorems 3.6 and 3.3, we have $\mathfrak{B}^- = X\mathfrak{M}^+$ and $\mathfrak{B}^+ = X\mathfrak{M}^-$, where X is an isometric transformation with domain $\mathfrak{M}$ and range $\mathfrak{B}^+ + \mathfrak{B}^-$. Thus $\mathfrak{B}^-$ has the dimension number m, $\mathfrak{B}^+$ the dimension number n. Since $\mathfrak{B}^-$ and $\mathfrak{B}^+$ clearly have the same dimension numbers as $\mathfrak{D}^-$ and $\mathfrak{D}^+$, respectively, the deficiency index of H is (n, m).

To prove the second assertion of the theorem, we observe first that a reduction operator A is unbounded if and only if $B_1 \equiv (A_1^*A_1)^{1/2}$ is unbounded. Now let $\mathfrak{N}$ be the range of $E_A(1-0)$. Then, by Theorem 3.5, the range of

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† Murray, Theorem 1.24.

$I - E_A(1)$ is $Q\mathfrak{N}$ and $\mathfrak{N}$ is Q-symmetric. Furthermore, on $(\mathfrak{B}^+ + \mathfrak{B}^-) \ominus (\mathfrak{N} + Q\mathfrak{N})$, $B_1 = I$. Thus B_1 induces a bounded transformation in $(\mathfrak{B}^+ + \mathfrak{B}^-) \ominus Q\mathfrak{N}$, and so is unbounded if and only if it induces an unbounded transformation in $Q\mathfrak{N}$. Since it is evident that B_1 can be constructed with this property if and only if $\mathfrak{N}$ has the dimension number $\aleph_0$, and since, by Theorem 2.2, $\mathfrak{N}$ can be chosen with that dimension number if and only if both of the characteristic manifolds of Q are Hilbert spaces, the proof is complete.

3. **The graph of a reduction operator.** For further study of reduction operators, we now call into play Theorem 2.11. We preserve the meanings of all the symbols introduced in that theorem.

THEOREM 3.8. *The space $\mathfrak{B} + \mathfrak{D}$ in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ is U-symmetric and $\mathfrak{B} + \mathfrak{D} \subseteq \mathfrak{B}(A)$, the equality holding if and only if $H \equiv H^*$. Let Z be the isometric transformation with domain $\mathfrak{L}^+$ and range $\mathfrak{L}^-$ associated with $\mathfrak{B} + \mathfrak{D}$ by Theorem 2.2, and let Y be the unitary transformation of $\mathfrak{L}^+$ into $\mathfrak{L}^-$ associated with $\mathfrak{B}(A)$ in accordance with Theorems 2.11 and 2.2. Then $Z \subseteq Y$, the equality holding if and only if $H \equiv H^*$. The manifolds $\mathfrak{L}^+$ and $\mathfrak{L}^-$ admit the decompositions*

(1) $$\mathfrak{L}^+ = \mathfrak{D}(Z) \oplus \mathfrak{B}^+ \oplus \mathfrak{M}^+,$$

(2) $$\mathfrak{L}^- = \mathfrak{R}(Z) \oplus \mathfrak{B}^- \oplus \mathfrak{M}^-,$$

where the three component manifolds in each case are mutually orthogonal, and $\mathfrak{B}^- \oplus \mathfrak{M}^- = Y(\mathfrak{B}^+ \oplus \mathfrak{M}^+)$.

That the relation $\mathfrak{B} + \mathfrak{D} \subseteq \mathfrak{B}(A)$ holds is obvious, since $\mathfrak{B}$ is the manifold of zeros of A. Hence, by Theorem 2.2, $Z \subseteq Y$, and $\mathfrak{B} + \mathfrak{D} = \mathfrak{D}(A)$ if and only if $Z \equiv Y$. But if $\mathfrak{B} + \mathfrak{D} = \mathfrak{D}(A)$, then $A \equiv O$, and conversely. Since $A \equiv O$ if and only if $H \equiv H^*$, the first two assertions of the theorem are established.

As we observed in Theorem 2.11, the manifold $\mathfrak{L}^+$ consists of all vectors in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ of the form $\{f, if, h_+\}$, where h_+ is in $\mathfrak{M}^+$, and $\mathfrak{L}^-$ of all vectors of the form $\{f, -if, h_-\}$, where h_- is in $\mathfrak{M}^-$. Since, if

$$X \equiv (H - iI)(H + iI)^{-1},$$

we have $\mathfrak{D}(X) = \mathfrak{H} \ominus \mathfrak{D}^+$ and $\mathfrak{R}(X) = \mathfrak{H} \ominus \mathfrak{D}^-$, it is readily shown, by reason of the relation between X and Z made evident by Theorem 2.7, that $\mathfrak{B}^+ \oplus \mathfrak{M}^+ = \mathfrak{L}^+ \ominus \mathfrak{D}(Z)$ and that $\mathfrak{B}^- \oplus \mathfrak{M}^- = \mathfrak{L}^- \ominus \mathfrak{R}(Z)$. From these equations the assertions of the theorem concerning the decomposition of $\mathfrak{L}^+$ and $\mathfrak{L}^-$ follow.

Finally, the equation $\mathfrak{B}^- \oplus \mathfrak{M}^- = Y(\mathfrak{B}^+ \oplus \mathfrak{M}^+)$ is a consequence of the unitary character of Y and the facts already established.

THEOREM 3.9. *Let $\mathfrak{B}^+(A)$, $\mathfrak{B}^-(A)$, $\mathfrak{B}_+(A)$, and $\mathfrak{B}_-(A)$ be the subsets of $\mathfrak{B}(A_1)$ defined as follows: $\{f, H^*f, Af\}$ in $\mathfrak{B}(A_1)$ is in $\mathfrak{B}^+(A)$ if f is in $\mathfrak{D}^+$, and*

in $\mathfrak{B}^-(A)$ if f is in $\mathfrak{D}^-$; $\{f, H^*f, Af\}$ is in $\mathfrak{B}_+(A)$ if Af is in $\mathfrak{M}^+$, and in $\mathfrak{B}_-(A)$ if Af is in $\mathfrak{M}^-$. Then

(1) $$\mathfrak{B}^+(A) = (I - Y^{-1})\mathfrak{M}^-,$$

(2) $$\mathfrak{B}^-(A) = (I - Y)\mathfrak{M}^+,$$

(3) $$\mathfrak{B}_+(A) = (I - Y^{-1})\mathfrak{B}^-,$$

(4) $$\mathfrak{B}_-(A) = (I - Y)\mathfrak{B}^+;$$

and

(5) $$\mathfrak{B}(A_1) \ominus \mathfrak{B}^+(A) = \mathfrak{B}_+(A),$$

(6) $$\mathfrak{B}(A_1) \ominus \mathfrak{B}^-(A) = \mathfrak{B}_-(A).$$

Thus

(7) $$\mathfrak{B}(A) = \mathfrak{B} + \mathfrak{B}^+(A) + \mathfrak{B}_+(A) = \mathfrak{B} + \mathfrak{B}^-(A) + \mathfrak{B}_-(A).$$

From Theorem 3.8 it is evident that every element of $\mathfrak{B}(A_1)$ can be written in the form $(I - Y)\{f^+, if^+, h_+\}$ where f^+ is in $\mathfrak{D}^+$ and h_+ is in $\mathfrak{M}^+$, or equivalently in the form $(I - Y^{-1})\{f^-, -if^-, h_-\}$ where f^- is in $\mathfrak{D}^-$, h_- in $\mathfrak{M}^-$. However, since Y^{-1} takes $\mathfrak{B}^- + \mathfrak{M}^-$ into $\mathfrak{B}^+ + \mathfrak{M}^+$, it is clear that the projection of $(I - Y^{-1})\{f^-, -if^-, h_-\}$ on $\mathfrak{H} \oplus \mathfrak{H}$ is in $\mathfrak{B}^+$; that is, that $(I - Y^{-1})\{f^-, -if^-, h_-\}$ is in $\mathfrak{B}^+(A))$ if and only if $f^- = 0$. Thus (1) is valid. Equations (2), (3), and (4) can be verified by entirely similar arguments. Equation (5) then follows from equations (1) and (3) and equation (2) of Theorem 3.8, while (6) follows from (2) and (4) and equation (1) of Theorem 3.8. The relations (7) are consequences of (5) and (6) and the fact that $\mathfrak{B}(A_1) = \mathfrak{B}(A) \ominus (\mathfrak{B} + \mathfrak{D})$.

It is worth while to point out here a second characterization of reduction operators which is suggested by Theorems 2.11, 3.8, and 3.9. The facts can be stated as follows: Let $\mathfrak{H}$ be a Hilbert space, H a closed linear symmetric transformation in $\mathfrak{H}$. Let Q be the transformation in $\mathfrak{H} \oplus \mathfrak{H}$ which takes $\{f, g\}$ into $\{g, -f\}$, and let Z be the isometric transformation corresponding to the Q-symmetric manifold $\mathfrak{B} = \mathfrak{B}(H)$ in accordance with Theorem 2.2. Let $\mathfrak{M}^+$ and $\mathfrak{M}^-$ be complex euclidean spaces with the same dimension numbers as $\mathfrak{B}^-$ and $\mathfrak{B}^+$, respectively. Let Y be an arbitrary isometric transformation with domain $\mathfrak{D}(Z) + \mathfrak{B}^+ + \mathfrak{M}^+$ and range $\mathfrak{R}(Z) + \mathfrak{B}^- + \mathfrak{M}^-$ such that $Y \supseteq Z$, such that $E_{\mathfrak{B}} - Y h^+ = 0$ for h^+ in $\mathfrak{M}^+$ implies $h^+ = 0$, and such that

$$E_{\mathfrak{M}} - Y\{f^+, if^+\} = 0$$

for $\{f^+, if^+\}$ in $\mathfrak{B}^+$ implies $f^+ = 0$. Then $\mathfrak{R}(I - Y)$ is the graph of a reduction operator A for H^*. We leave the proof of these assertions to the reader.

Theorem 3.10. *Let* $\mathfrak{B}_A{}^+ = \mathfrak{B}^+ \cdot \mathfrak{D}(A)$, $\mathfrak{B}_A{}^- = \mathfrak{B}^- \cdot \mathfrak{D}(A)$, $\mathfrak{M}_A{}^+ = \mathfrak{M}^+ \cdot \mathfrak{R}(A)$, $\mathfrak{M}_A{}^- = \mathfrak{M}^- \cdot \mathfrak{R}(A)$. *Then* $\mathfrak{B}_A{}^+$, $\mathfrak{B}_A{}^-$, $\mathfrak{M}_A{}^+$, *and* $\mathfrak{M}_A{}^-$ *are linear manifolds everywhere dense in* $\mathfrak{B}^+$, $\mathfrak{B}^-$, $\mathfrak{M}^+$, *and* $\mathfrak{M}^-$, *respectively. Each of the equations* $\mathfrak{B}_A{}^+ = \mathfrak{B}^+$, $\mathfrak{B}_A{}^- = \mathfrak{B}^-$, $\mathfrak{M}_A{}^+ = \mathfrak{M}^+$, $\mathfrak{M}_A{}^- = \mathfrak{M}^-$, *is valid if and only if* A *is bounded. If* V *is the isometric transformation with domain* $\mathfrak{M}$ *and range* $\mathfrak{B}^+ + \mathfrak{B}^-$ *such that* $(A_1{}^*A_1)^{1/2} \equiv V A_1$, *then* $\mathfrak{B}_A{}^- = V\mathfrak{M}_A{}^+$, $\mathfrak{B}_A{}^+ = V\mathfrak{M}_A{}^-$.

The manifolds $\mathfrak{B}_A{}^+$, $\mathfrak{B}_A{}^-$, $\mathfrak{M}_A{}^+$, *and* $\mathfrak{M}_A{}^-$ *are characterized by the equations*

(1) $$\mathfrak{B}_A{}^+ = E_{\mathfrak{H} \oplus \mathfrak{H}}\mathfrak{B}^+(A) = E_{\mathfrak{H} \oplus \mathfrak{H}} Y^{-1}\mathfrak{M}^-,$$

(2) $$\mathfrak{B}_A{}^- = E_{\mathfrak{H} \oplus \mathfrak{H}}\mathfrak{B}^-(A) = E_{\mathfrak{H} \oplus \mathfrak{H}} Y\mathfrak{M}^+,$$

(3) $$\mathfrak{M}_A{}^+ = E_{\mathfrak{M}}\mathfrak{B}_+(A) = E_{\mathfrak{M}} Y^{-1}\mathfrak{B}^-,$$

(4) $$\mathfrak{M}_A{}^- = E_{\mathfrak{M}}\mathfrak{B}_-(A) = E_{\mathfrak{M}} Y\mathfrak{B}^+,$$

where each of the projections $E_{\mathfrak{H} \oplus \mathfrak{H}}$ *and* $E_{\mathfrak{M}}$ *has domain* $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$, *and* Y *has the meaning assigned to it in Theorem* 3.8.

Again we denote by $\mathfrak{N}$ the range of $E_A(1-0)$, and by $\mathfrak{P}$ the range of $E_A(1) - E_A(1-0)$. Then $\mathfrak{N}$ and $\mathfrak{P}$ both reduce $B_1 \equiv (A_1{}^*A_1)^{1/2}$, and in $\mathfrak{P}$, $B_1 = I$, while in $\mathfrak{N}$, B_1 induces a bounded self-adjoint transformation. Furthermore, by Theorem 3.5, $\mathfrak{P} = \mathfrak{B}_1{}^+ + \mathfrak{B}_1{}^-$, where $\mathfrak{B}_1{}^+$ and $\mathfrak{B}_1{}^-$ are closed linear manifolds in $\mathfrak{B}^+$ and $\mathfrak{B}^-$, respectively; in addition, applying the same theorem and Theorem 2.2, we have $\mathfrak{P} = (\mathfrak{B}^+ + \mathfrak{B}^-) \ominus (\mathfrak{N} + Q\mathfrak{N})$ and $\mathfrak{N} + Q\mathfrak{N} = \mathfrak{D}(X) + \mathfrak{R}(X)$, where X is an isometric transformation and $\mathfrak{D}(X) = \mathfrak{B}^+ \ominus \mathfrak{B}_1{}^+$, $\mathfrak{R}(X) = \mathfrak{B}^- \ominus \mathfrak{B}_1{}^-$, $\mathfrak{R}(I - X) = \mathfrak{N}$.

Now let $\mathfrak{N}_1$ be the range of the transformation B_2 induced in $\mathfrak{N}$ by B_1. Then, since B_2 is self-adjoint and has an inverse, $\mathfrak{N}_1$ is a linear manifold dense in $\mathfrak{N}$, and $\mathfrak{D}(A_1) = \mathfrak{D}(B_1) = \mathfrak{P} + \mathfrak{N} + Q\mathfrak{N}_1$. Hence

$$\mathfrak{D}(A) \cdot \mathfrak{B}^+ = \mathfrak{D}(A_1) \cdot \mathfrak{B}^+ = \mathfrak{B}_1{}^+ + (\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^+,$$

$$\mathfrak{D}(A) \cdot \mathfrak{B}^- = \mathfrak{D}(A_1) \cdot \mathfrak{B}^- = \mathfrak{B}_1{}^- + (\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^-.$$

Consequently, $\mathfrak{B}_A{}^+$ ($\mathfrak{B}_A{}^-$) is dense in $\mathfrak{B}^+$ ($\mathfrak{B}^-$) if and only if $(\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^+$ $((\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^-)$ is dense in $\mathfrak{D}(X)$ ($\mathfrak{R}(X)$).

To prove that the latter condition is satisfied, we denote by X_1 the isometric transformation from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that $\mathfrak{N}_1 = \mathfrak{R}(I - X_1)$. Then evidently $Q\mathfrak{N}_1 = \mathfrak{R}(I + X_1)$. Thus, since $\mathfrak{N} \supseteq \mathfrak{R}(I - X_1)$, we have

$$(\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^+ = \mathfrak{D}(X_1), \qquad (\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^- = \mathfrak{R}(X_1),$$

and

$$\mathfrak{B}_A{}^+ = \mathfrak{B}_1{}^+ + \mathfrak{D}(X_1), \qquad \mathfrak{B}_A{}^- = \mathfrak{B}_1{}^- + \mathfrak{R}(X_1).$$

But, since the closure of $\mathfrak{N}_1$ is $\mathfrak{N}$, it follows from Theorem 2.2 that $\bar{X}_1 \equiv X$,

and therefore that the closures of the domain and range of X_1 are, respectively, the domain and range of X. Furthermore, $\mathfrak{N}_1 = \mathfrak{N}$ if and only if B_2 has a bounded inverse; that is, if and only if A is bounded. Thus $X_1 \equiv X$ if and only if A is bounded. Therefore $\mathfrak{B}_A{}^+$ and $\mathfrak{B}_A{}^-$ are linear manifolds everywhere dense in $\mathfrak{B}^+$ and $\mathfrak{B}^-$, respectively, and $\mathfrak{B}_A{}^+ = \mathfrak{B}^+$, $\mathfrak{B}_A{}^- = \mathfrak{B}^-$ if and only if A is bounded.

We now observe that $\mathfrak{R}(B_1) = V\mathfrak{R}(A_1)$ and, since $(\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^+ = (\mathfrak{N}_1 + Q\mathfrak{N}) \cdot \mathfrak{B}^+$ and $(\mathfrak{N} + Q\mathfrak{N}_1) \cdot \mathfrak{B}^- = (\mathfrak{N}_1 + Q\mathfrak{N}) \cdot \mathfrak{B}^-$, that

$$\mathfrak{D}(B_1) \cdot \mathfrak{B}^+ = \mathfrak{R}(B_1) \cdot \mathfrak{B}^-, \qquad \mathfrak{D}(B_1) \cdot \mathfrak{B}^- = \mathfrak{R}(B_1) \cdot \mathfrak{B}^-.$$

Consequently, taking account of the relations $\mathfrak{B}^- = V\mathfrak{M}^+$, $\mathfrak{B}^+ = V\mathfrak{M}^-$, we have $\mathfrak{B}_A{}^+ = V\mathfrak{M}_A{}^-$, $\mathfrak{B}_A{}^- = V\mathfrak{M}_A{}^+$. Thus, since V is isometric, it follows that $\mathfrak{M}_A{}^+$ and $\mathfrak{M}_A{}^-$ are dense in $\mathfrak{M}^+$ and $\mathfrak{M}^-$, respectively, and that $\mathfrak{M}_A{}^+ = \mathfrak{M}^+$, $\mathfrak{M}_A{}^- = \mathfrak{M}^-$ if and only if A is bounded. Thus the facts stated in the first paragraph are all established.

The equations with which the theorem concludes are consequences of Theorems 2.11, 3.8, and 3.9, as the reader can readily verify.

THEOREM 3.11. *Let A be a reduction operator for H^*, and let Y have the same meaning as in Theorem 3.8. Let F be the transformation with domain $\mathfrak{M}^+$ which is equal on its domain to $E_{\mathfrak{M}}Y$, and let G be the transformation with domain $\mathfrak{B}^+$ which is equal on its domain to $E_{\mathfrak{H} \oplus \mathfrak{H}}Y$. Then F and G are bounded linear transformations with ranges in $\mathfrak{M}^-$ and $\mathfrak{B}^-$, respectively. The adjoints F^* and G^*, with the respective domains $\mathfrak{M}^-$ and $\mathfrak{B}^-$, are equal on their domains to $E_{\mathfrak{M}}Y^{-1}$ and $E_{\mathfrak{H} \oplus \mathfrak{H}}Y^{-1}$, respectively. If A is bounded, F, G, F^*, and G^* each have as bounds constants less than 1; if A is unbounded, F, G, F^*, and G^* each have the bound 1, but none of these transformations attains its bound. Finally,*

(1) $$\mathfrak{R}(I - F) = A_1 \mathfrak{B}_A{}^- = E_{\mathfrak{M}} \mathfrak{B}^-(A),$$

(2) $$\mathfrak{R}(I - F^*) = A_1 \mathfrak{B}_A{}^+ = E_{\mathfrak{M}} \mathfrak{B}^+(A),$$

(3) $$\mathfrak{R}(I - G) = A_1^{-1} \mathfrak{M}_A{}^- = E_{\mathfrak{H} \oplus \mathfrak{H}} \mathfrak{B}_-(A),$$

(4) $$\mathfrak{R}(I - G^*) = A_1^{-1} \mathfrak{M}_A{}^+ = E_{\mathfrak{H} \oplus \mathfrak{H}} \mathfrak{B}_+(A).$$

That F and G are bounded linear with bounds less than or equal to unity is an immediate consequence of their definitions in terms of the isometric transformation Y. That their ranges are in $\mathfrak{M}^-$ and $\mathfrak{B}^-$, respectively, follows from the equation $\mathfrak{B}^- + \mathfrak{M}^- = Y(\mathfrak{B}^+ + \mathfrak{M}^+)$ of Theorem 3.8. Since the adjoint from $\mathfrak{L}^+$ to $\mathfrak{L}^-$ of Y is Y^{-1} and since $E_{\mathfrak{H} \oplus \mathfrak{H}}Y^{-1}\mathfrak{M}^-$ is orthogonal to $\mathfrak{M}^+$, while $E_{\mathfrak{M}}Y^{-1}\mathfrak{B}^-$ is orthogonal to $\mathfrak{B}^+$, we have at once that F^* and G^* are equal on their domains to $E_{\mathfrak{M}}Y^{-1}$ and $E_{\mathfrak{H} \oplus \mathfrak{H}}Y^{-1}$, respectively.

Now consider all elements of $\mathfrak{B}(A) = \mathfrak{R}(I - Y)$ which are of the form $\{0, 0, h^+\} - Y\{0, 0, h^+\}$, where h^+ is in $\mathfrak{M}^+$. By Theorem 3.8, every such element can be written in the form $\{-f^-, if^-, h^+ - h^-\}$, where $\{f^-, -if^-\}$ is in $\mathfrak{B}^-$, h^- in $\mathfrak{M}^-$. Conversely, every element of $\mathfrak{B}^-(A)$ can be written in the form $\{0, 0, h^+\} - Y\{0, 0, h^+\}$, where h^+ is in $\mathfrak{M}^+$. Furthermore, we have $h^+ - h^- = -A\{f^-, -if^-\}$, $h^- = Fh^+$. Let us suppose, then, that the bound of F is 1. Then we can choose a sequence $\{h_n^+\}$ in $\mathfrak{M}^+$ so that $|h_n^+| = 1$, $|h_n^-| \to 1$, $n \to \infty$, where $h_n^- = Fh_n^+$. But, if $\{f_n^-, -if_n^-, h_n^-\} = Yh_n^+$, this implies that $\lim_{n \to \infty}|\{f_n^-, -if_n^-\}| = 0$. Consequently, since $h_n^+ - h_n^- = A\{f_n^-, -if_n^-\}$ and $\lim_{n \to \infty}|h_n^+ - h_n^-| = 2^{1/2}$, it follows that A is unbounded.

Now let us suppose that A is unbounded. Then, by Theorem 3.9, $\mathfrak{B}_A^-$ is a linear manifold dense in $\mathfrak{B}^-$, $\mathfrak{B}_A^- \neq \mathfrak{B}^-$. Therefore the bounded linear transformation with domain $\mathfrak{M}^+$ which is equal on its domain to $E_{\mathfrak{H} \oplus \mathfrak{H}} Y$ has an unbounded inverse, since otherwise its range, which according to Theorem 3.9 is $\mathfrak{B}_A^-$, would be closed. Consequently, we can choose a sequence $\{\{f_n^-, -if_n^-\}\}$ in $\mathfrak{B}_A^-$ such that

$$\lim_{n \to \infty} |\{f_n^-, -if_n^-\}| = 0, \qquad |E_{\mathfrak{M}^+}A\{f_n^-, -if_n^-\}| = 1.$$

Let $E_M^+A\{f_n^-, -if_n^-\} = h_n^+$. Then

$$|h_n^+|^2 = |\{f_n^-, if_n^-\}|^2 + |Fh_n^+|^2.$$

Thus $\lim_{n \to \infty}|Fh_n^+| = 1$, and F has the bound 1.

The analogous facts concerning G are readily established by entirely similar arguments making use of the transformation A_1^{-1} and the fact that A_1^{-1} and A are bounded or unbounded together; we omit the details.

Since F^* and G^* evidently have the same bounds as F and G, respectively, it follows that F^* and G^* have the bound 1 if and only if A is unbounded.

Now suppose that $|Fh^+| = |E_{\mathfrak{M}}Yh^+| = |h^+|$ for some h^+ in $\mathfrak{M}^+$. Then, since Y is unitary, $E_{\mathfrak{H} \oplus \mathfrak{H}}Yh^+ = 0$, which clearly implies $h^+ = 0$. Hence F never attains the bound 1. Similar arguments serve to establish that F^*, G, and G^* never attain the bound 1.

The formulas with which the theorem concludes are readily verified and we omit detailed proofs.

4. **Two types of reduction operator.** We have previously had occasion to distinguish between bounded and unbounded reduction operators. For reasons which will be made clear later, it is necessary for us now to distinguish two distinct types of unbounded reduction operator. Furthermore, although it is not necessary, it is clarifying also to distinguish two corresponding types of bounded reduction operator. Hence we introduce the following definition:

DEFINITION 3.5. *A reduction operator A for H^* is said to be of type I if at least one of the manifolds $\mathfrak{B}_A{}^+$, $\mathfrak{B}_A{}^-$ of Theorem 3.9 contains no closed linear manifold with the dimension number $\aleph_0$. Otherwise A is said to be of type II.*

THEOREM 3.12. *A bounded reduction operator A is of type I if and only if one of the manifolds $\mathfrak{B}^+$, $\mathfrak{B}^-$ is a unitary space.*

Theorem 3.12 is an immediate consequence of the fact that when A is bounded we have, according to Theorem 3.10, $\mathfrak{B}_A{}^+ = \mathfrak{B}^+$, $\mathfrak{B}_A{}^- = \mathfrak{B}^-$.

In our investigation of the significance of Definition 3.5 for unbounded reduction operators, the following lemma plays an essential role.

LEMMA 3.1. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces, and let T be a closed linear transformation with domain dense in $\mathfrak{H}_1$ and range in $\mathfrak{H}_2$. Then a necessary and sufficient condition that T be totally continuous is that each closed linear manifold in its range have a finite dimension number.*

We first take account of the fact that the transformation T is totally continuous if and only if T^* is,[†] the existence of T^* being assured by the hypothesis that $\mathfrak{D}_{(T)}$ is dense in $\mathfrak{H}$. But $T^* \equiv V(TT^*)^{1/2}$, where V is isometric; thus T is totally continuous if and only if the nonnegative definite self-adjoint transformation $(TT^*)^{1/2}$ in $\mathfrak{H}_2$ is totally continuous; furthermore, $(TT^*)^{1/2}$ has the same range as T.[‡] Consequently, it is sufficient to prove the lemma for the case that $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ and T is a nonnegative definite self-adjoint transformation in $\mathfrak{H}$; we now restrict our attention to this case.

But even further simplification is possible. For if $\mathfrak{N}$ denotes the manifold of zeros of T, $\mathfrak{H} \ominus \mathfrak{N}$ reduces T; and, in $\mathfrak{H} \ominus \mathfrak{N}$, T induces a nonnegative definite self-adjoint transformation T_1 whose inverse exists; moreover, T_1 which has the same range as T, and is totally continuous if and only if T is. Consequently, since the trivial case that $\mathfrak{H} \ominus \mathfrak{N}$ is a unitary space is readily disposed of, it is permissible to assume that the nonnegative definite transformation T in $\mathfrak{H}$ has an inverse.

We now prove the necessity of the condition for the specialization of the lemma which we have shown to be equivalent to the general statement. Let us suppose that T is totally continuous and that $\mathfrak{M}$ is a closed linear manifold in the range of T. Let $\mathfrak{N}$ be the closed linear manifold determined by the set into which T^{-1} takes $\mathfrak{M}$, and observe that since T is closed and totally continuous, it is bounded and defined throughout $\mathfrak{H}$. Hence the transformation T_0 with domain $\mathfrak{N} \cdot \mathfrak{D}(T)$, which is equal on its domain to T, is a bounded linear transformation with domain $\mathfrak{N}$ and range $\mathfrak{M}$. Furthermore T_0^{-1} exists

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† Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 100, Theorem 5.
‡ Murray, Theorem 1.24.

and is closed. Hence, since its domain is the entire space $\mathfrak{M}$, T_0^{-1} is bounded.[†] Consequently, if $\mathfrak{U}$ is a bounded set in $\mathfrak{M}$, $T_0^{-1}\mathfrak{U}$ is a bounded set in $\mathfrak{N}$. But, since T is totally continuous, this implies that $\mathfrak{U}$ is a compact set in $\mathfrak{M}$. Therefore every bounded set in $\mathfrak{M}$ is compact, from which it follows that $\mathfrak{M}$ has a finite dimension number.

[N]ow let us suppose that no closed linear manifold in $\mathfrak{R}(T)$ has the dimension number $\aleph_0$. Let $E(\lambda)$ be the resolution of the identity associated with T, and let $\{\lambda_n\}$ be a monotone decreasing sequence of positive real numbers convergent to zero and having no member belonging to the point spectrum of T. Let $E(\Delta_1) = I - E(\lambda_1)$, $E(\Delta_n) = E(\lambda_{n-1}) - E(\lambda_n)$, $n = 2, 3, 4, \cdots$. Then the range $\mathfrak{M}_n$ of $E(\Delta_n)$ reduces T, and T induces in $\mathfrak{M}_n$ a transformation T_n with bounded inverse. Moreover, T_n^{-1} is closed and therefore is defined throughout $\mathfrak{M}_n$. Hence $\mathfrak{M}_n$ belongs to the range of T and, in consequence, is a unitary space. It follows that the points of the spectrum of T on the interval $\lambda_{n-1} \leq \lambda \leq \lambda_n$ are finite in number and are all characteristic values (the characteristic values of T_n) and that each such characteristic value has finite multiplicity. Thus the characteristic values of T all have finite multiplicity and can be arranged in a sequence which is bounded and has, as its only possible limit point, the origin, while the continuous spectrum of T is either empty or contains only the origin. (Here, since T^{-1} exists, the latter alternative must hold and the origin must be, in fact, a limit point of the spectrum.) Since these spectral properties of T, deduced from the hypothesis that its range contains no Hilbert spaces, are known to be sufficient for the total continuity of T,[‡] the lemma stands established.

THEOREM 3.13. *Let A be an unbounded reduction operator, and let $B_1 \equiv (A_1^* A_1)^{1/2}$. Let $\mathfrak{P}$ be the manifold on which $B_1 = I$. Then for A to be of type* I *it is necessary and sufficient that both of the following conditions be satisfied:*

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† Murray, Theorem 1.25.

‡ The theorem which we use here can be stated briefly as follows: If the characteristic values other than zero of a self-adjoint transformation T each have finite multiplicity and are either finite in number or form a bounded set with zero as its only limit point and if the continuous spectrum of T contains no point except possibly the origin, then T is totally continuous. We find no proof of this result in the literature, although the theorem has been stated and used by several writers. It may not be amiss, therefore, to point out here that the theorem follows almost immediately from a theorem of S. Banach (*Théorie des Opérations Linéaires*, Warsaw, 1932, p. 96, Theorem 2). The latter states that the class of all totally continuous transformations T in a Banach space is closed in the topology defined by setting $|T_1 - T_2|$ equal to the bound of $T_1 - T_2$. Since the spectral properties of T described in the theorem in question permit us to conclude that T either has a range with a finite dimension number or is the limit in the sense of the above topology of a sequence of transformations each with that property, and since every linear transformation whose range has a finite dimension number is obviously totally continuous, no further argument is required. Another proof of the theorem, which does not make use of Banach's result, has been communicated to the author by B. Lengyel.

(1) B_1 *induces a totally continuous transformation in the range of* $E_A(1-0)$;

(2) *at least one of the manifolds* $\mathfrak{B}^+ \cdot \mathfrak{P}$ *and* $\mathfrak{B}^- \cdot \mathfrak{P}$ *is a unitary space.*

Since both of the manifolds $\mathfrak{B}^+ \cdot \mathfrak{P}$ and $\mathfrak{B}^- \cdot \mathfrak{P}$ belong to $\mathfrak{D}(B_1) = \mathfrak{D}(A_1)$, the necessity of (2) is evident. To prove the necessity of (1), we start with the assumption that the transformation B_0, induced in the range $\mathfrak{N}$ of $E_A(1-0)$ by B_1, is not totally continuous. Then, by Lemma 3.1, there is, in the range of B_0, a closed linear manifold $\mathfrak{N}_1$ with the dimension number $\aleph_0$. Thus $\mathfrak{N}_1$ is in the domain of B_0^{-1}. Therefore, by Theorems 3.5 and 3.6, $Q\mathfrak{N}_1$ is in the domain of B_1, where $Q = iI$ on $\mathfrak{B}^+$, $Q = -iI$ on $\mathfrak{B}^-$. Consequently, $\mathfrak{N}_1 + Q\mathfrak{N}_1$ is in the domain of B_1. But since $\mathfrak{N}_1$ is Q-symmetric and closed, $\mathfrak{N}_1 = \mathfrak{R}(I - X)$, $Q\mathfrak{N}_1 = \mathfrak{R}(I + X)$, where X is closed and isometric with domain in $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$, by Theorem 2.2. Hence $\mathfrak{D}(X)$ and $\mathfrak{R}(X)$ both belong to the domain of B_1, and both of these manifolds are Hilbert spaces because $\mathfrak{N}_1$ is a Hilbert space. Hence, if B_0 is not totally continuous, A is of type II, and therefore (1) is necessary as we wished to prove.

We prove now the sufficiency of the two conditions. We assume first that $\mathfrak{B}^+ \cdot \mathfrak{P}$ is a unitary space and denote by $\mathfrak{N}_0$ the range of the transformation B_0 introduced above. Then, by applying Theorems 3.5 and 3.6, we can resolve the domain of A_1 according to the equation

$$\mathfrak{D}(A_1) = \mathfrak{D}(B_1) = \mathfrak{N} + Q\mathfrak{N}_0 + \mathfrak{B}^+ \cdot \mathfrak{P} + \mathfrak{B}^- \cdot \mathfrak{P}.$$

Now let X be the isometric transformation associated with $\mathfrak{N}$ by Theorem 2.2, X_0 the isometric transformation associated with $\mathfrak{N}_0$. Then $\tilde{X}_0 \equiv X$. Furthermore, $\mathfrak{D}(X_0)$ contains no Hilbert spaces if $\mathfrak{N}_0$ contains no Hilbert spaces and thus, according to Lemma 3.1, if B_0 is totally continuous. But $\mathfrak{D}(A_1) \cdot \mathfrak{B}^+ = (\mathfrak{N} + Q\mathfrak{N}_0) \cdot \mathfrak{B}^+ + \mathfrak{P} \cdot \mathfrak{B}^+$, and thus $\mathfrak{B}_A{}^+ = \mathfrak{D}(A_1) \cdot \mathfrak{B}^+ = \mathfrak{D}(X_0) + \mathfrak{P} \cdot \mathfrak{B}^+$. Consequently, if $\mathfrak{P} \cdot \mathfrak{B}^+$ is a unitary space and (1) is satisfied, $\mathfrak{B}_A{}^+$ contains no Hilbert spaces. On the other hand, if $\mathfrak{P} \cdot \mathfrak{B}^-$ contains no closed linear manifold with the dimension number $\aleph_0$ and (1) is satisfied, an entirely similar argument leads to the conclusion that $\mathfrak{B}_A{}^-$ contains no closed linear manifold with the dimension number $\aleph_0$. Thus (1) and (2) are sufficient for A to be of type I and the proof is complete.

THEOREM 3.14. *Let* Y *have the same meaning as in Theorem 3.8,* $\mathfrak{M}_A{}^+$ *and* $\mathfrak{M}_A{}^-$ *the same meaning as in Theorem 3.10. Then for a reduction operator* A *to be of type* I, *any of the following conditions is necessary and sufficient:*

(1) *At least one of the manifolds* $\mathfrak{M}_A{}^+$, $\mathfrak{M}_A{}^-$ *contains no closed linear manifold with the dimension number* $\aleph_0$.

(2) *Either the transformation with domain* $\mathfrak{M}^+$ *which is equal on its domain to* $E_{\mathfrak{B}} - Y$ *or the transformation with domain* $\mathfrak{M}^-$ *which is equal on its domain to* $E_{\mathfrak{B}} + Y^{-1}$ *is totally continuous.*

(3) *Either the transformation with domain $\mathfrak{B}^+$ which is equal on its domain to $E_{\mathfrak{M}}-Y$ or the transformation with domain $\mathfrak{B}^-$ which is equal on its domain to $E_{\mathfrak{M}}+Y^{-1}$ is totally continuous.*

By Theorem 3.10, the two transformations defined under (2) have ranges $\mathfrak{B}_A{}^-$ and $\mathfrak{B}_A{}^+$, respectively. Thus the necessity and sufficiency of (2) follow at once from Lemma 3.1. Furthermore, since the two transformations defined under (3) have ranges, respectively, $\mathfrak{M}_A{}^-$ and $\mathfrak{M}_A{}^+$, the same lemma serves to establish that the conditions (1) and (3) are coextensive. Thus we can complete the proof by showing that (1) is necessary and sufficient, and the latter follows at once when we recall from the proof of Theorem 3.11 that $\mathfrak{M}_A{}^- = V\mathfrak{B}_A{}^+$ and $\mathfrak{M}_A{}^+ = V\mathfrak{B}_A{}^-$, where V is isometric.

For use later, we introduce now the following definition:

DEFINITION 3.6. *Let A be a reduction operator of type* I, *and let $B_1 \equiv (A_1^*A_1)^{1/2}$. Let $\mathfrak{P}$ be the manifold on which $B_1 = I$, and let j and k be the dimension numbers of $\mathfrak{P} \cdot \mathfrak{B}^+$ and $\mathfrak{P} \cdot \mathfrak{B}^-$, respectively. Then, if j exceeds k, $(j-k, 0)$ is called the characteristic index of A. Otherwise, $(0, k-j)$ is called the characteristic index of A.*

THEOREM 3.15. *Let A be a bounded reduction operator of type* I, *and let m be the dimension number of $\mathfrak{M}^+$, n the dimension number of $\mathfrak{M}^-$. Then, if n exceeds m, $(n-m, 0)$ is the characteristic index of A; otherwise $(0, m-n)$ is the characteristic index of A.*

Let V be the isometric transformation with domain in $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$ associated with the range $\mathfrak{N}$ of $E_A(1-0)$ by Theorem 2.2. Then $\mathfrak{B}^+\ominus\mathfrak{D}(V) = \mathfrak{P} \cdot \mathfrak{B}^+$, $\mathfrak{B}^-\ominus\mathfrak{N}(V) = \mathfrak{P} \cdot \mathfrak{B}^-$, by Theorems 3.5 and 2.2, and $\mathfrak{D}(V)$ and $\mathfrak{N}(V)$ have the same dimension number. Furthermore, since A is bounded and of type I, at least one of the manifolds $\mathfrak{B}^+$, $\mathfrak{B}^-$ is a unitary space. Consequently, since $\mathfrak{B}^+$ and $\mathfrak{B}^-$ have the same dimension numbers as $\mathfrak{M}^+$ and $\mathfrak{M}^-$, respectively, we have at once $n-m=j-k$, where k and j have the same meanings as in Definition 3.6, and the theorem follows.

To conclude this section, we state without formal proof the following theorem:

THEOREM 3.16. *Let A and C be equivalent reduction operators for H^*. Then A and C are either both bounded or both unbounded and either both of type* I *with the same characteristic index, or both of type* II.

5. Self-adjoint extension by means of reduction operators.

A fundamental problem, with which the present work is connected, is the determination of all the self-adjoint extensions in a Hilbert space of a given symmetric transformation H in $\mathfrak{H}$. We define here a quite different type of self-adjoint exten-

sion which is obtained by extending the space $\mathfrak{H}$ and which is in a certain sense—not the usual one—an "extension" of H^* or of a transformation whose closure is H^*. Our construction, as well as the usual one mentioned first, is possible if and only if H has the deficiency index (n, n).

THEOREM 3.17. *Let H be a closed symmetric transformation in a Hilbert space $\mathfrak{H}$, and let H have the deficiency index (n, n), $(n > 0)$. Let $\mathfrak{L}$ be a complex euclidean space with the dimension number n. Let $\mathcal{C}$ be the class of all reduction operators A for H^* which have the following properties:*

(1) *the range-space of A is $\mathfrak{L} \oplus \mathfrak{L}$;*

(2) *the unitary transformation W in $\mathfrak{L} \oplus \mathfrak{L}$, associated with A by Definition 1.1, is that which takes $\{h, k\}$ into $\{k, -h\}$.*

Let S be the class of all self-adjoint transformations S in $\mathfrak{H} \oplus \mathfrak{L}$ which have the following properties:

(3) *if $\{f, h\}$ is in the domain of S and $S\{f, h\} = \{g, k\}$, then f is in the domain of H^* and $H^*f = g$;*

(4) *the set of elements $\{f, H^*f\}$ of $\mathfrak{B}^*$, such that $\{f, h\}$ is in $\mathfrak{D}(S)$ for some h, is dense in $\mathfrak{B}^*$;*

(5) *if $\{0, h\}$ is in the domain of S, then $h = 0$.*

Let X be the transformation which takes $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{L} \oplus \mathfrak{L}$ into $\mathfrak{H} \oplus \mathfrak{L} \oplus \mathfrak{H} \oplus \mathfrak{L}$ according to the equation

$$X\{f, g, h, k\} = \{f, h, g, k\}.$$

Then there is a one-to-one correspondence between $\mathcal{C}$ and S such that, when A and S correspond, $\mathfrak{B}(S) = X\mathfrak{B}(A)$.

Let A be an arbitrary member of $\mathcal{C}$, and let $\mathfrak{N} = X\mathfrak{B}(A)$. Then $\mathfrak{N}$ consists of all vectors $\{f, h, H^*f, k\}$ in $\mathfrak{H} \oplus \mathfrak{L} \oplus \mathfrak{H} \oplus \mathfrak{L}$ such that $\{f, H^*f\}$ is in $\mathfrak{D}(A)$ and $\{h, k\} = A\{f, H^*f\}$. Furthermore, f, and thus $\{f, h\}$, determines k uniquely, since f determines $\{h, k\}$ in $\mathfrak{L} \oplus \mathfrak{L}$ uniquely through the transformation A. Thus the transformation S whose domain is the set of all elements $\{f, h\}$ in $\mathfrak{H} \oplus \mathfrak{L}$ such that f is in $\mathfrak{D}^*$, $\{f, H^*f\}$ in $\mathfrak{D}(A)$, $\{h, k\} = A\{f, H^*f\}$ for some k in $\mathfrak{L}$, and which is defined by the equation $\{H^*f, k\} = S\{f, h\}$, is a one-valued transformation. Furthermore, by Definition 1.1, and condition (2) of the Theorem, S is self-adjoint, while by definition, S has the property (3). That S has the property (4) is a consequence of the fact that $\mathfrak{D}(A)$ is dense in $\mathfrak{B}^*$. That it has the property (5) follows from the fact that A is one-valued. Thus S belongs to S.

Now let S be an arbitrary member of S. Then $\mathfrak{B}(S)$ consists of a set of vectors of the form $\{f, h, H^*f, k\}$ in $\mathfrak{H} \oplus \mathfrak{L} \oplus \mathfrak{H} \oplus \mathfrak{L}$. Furthermore, according to (5), h and k are uniquely determined by f, and thus by $\{f, H^*f\}$. Conse-

quently the transformation A whose domain is the set of all elements $\{f, H^*f\}$ of $\mathfrak{B}^*$ such that for some h and k in $\mathfrak{L}$, $\{f, H, H^*f, k\}$ is in $\mathfrak{B}(S)$, and which is defined by the equation $A\{f, H^*f\} = \{h, k\}$, is a single-valued transformation. Furthermore, since S is linear, $\mathfrak{B}(S)$ is linear; thus $\mathfrak{B}(A) = X^{-1}\mathfrak{B}(S)$ is linear. Therefore A is linear. That A has domain dense in $\mathfrak{B}^*$ follows from (4); that it satisfies the other conditions of Definition 1.1 is an immediate consequence of the fact that S is self-adjoint. Since it is readily verified that A has the properties (1) and (2), A belongs to $\mathcal{C}$.

Finally, since the correspondence is described by the equation $\mathfrak{B}(S) = X\mathfrak{B}(A)$, it is clear that it is one-to-one.

We conclude with the observation, readily corroborated with the aid of Theorem 3.3, that if H has the deficiency index (n, n), $(n \neq 0)$, every equivalence class of reduction operators for H^* has in common with the class $\mathcal{C}$ of Theorem 3.17 a subclass whose cardinal number is that of the continuum.

CHAPTER IV. BOUNDARY CONDITIONS

1. **Introduction.** We are now prepared to discuss in detail the fundamental problem proposed in Chapter I, §3, and various questions which arise in connection with it.

As in Chapter III, we consider a fixed but arbitrary symmetric transformation H in $\mathfrak{H}$ and the adjoint H^* of H. We consider also a fixed but arbitrary reduction operator A for H^* and its contraction A_1 with domain $(\mathfrak{B}^+ + \mathfrak{B}^-) \cdot \mathfrak{D}(A)$. We preserve the meanings of all the symbols introduced in Definition 3.1 and adopt as standard the symbol $\mathfrak{M}$ for the range-space of A, the symbol W for the unitary transformation in $\mathfrak{M}$ associated with A by Definition 1.1, and the symbols $\mathfrak{M}^+$ and $\mathfrak{M}^-$ for the characteristic manifolds of W corresponding to the characteristic values $+i$ and $-i$, respectively. Finally, except where otherwise indicated, we continue to reserve the letter Q for the designation of the transformation of $\mathfrak{B}^+ + \mathfrak{B}^-$ into itself which is equal to iI on $\mathfrak{B}^+$, to $-iI$ on $\mathfrak{B}^-$.

2. **Boundary conditions involving a bounded reduction operator.** We first dispose of the case that $\mathfrak{M}$ is a unitary space and of the case that $\mathfrak{M}$ is a Hilber space with A bounded. As we have already pointed out, sufficient information for the solution of our problem in this case is provided by Theorem 3.2.

THEOREM 4.1. *Let $\mathfrak{M}$ be a unitary space, or let $\mathfrak{M}$ be a Hilbert space with A bounded. Then there is a one-to-one correspondence between the class of all closed linear symmetric extensions S of H and the class of all closed linear W-symmetric manifolds $\mathfrak{N}$ in $\mathfrak{M}$; S and $\mathfrak{N}$ correspond if $A\mathfrak{B}(S) = \mathfrak{N}$, or, in other words, if $S \equiv H(\mathfrak{N})$ in the sense of Definition 1.2. When S and $\mathfrak{N}$ correspond,*

$A\mathfrak{B}(S^*) = (\mathfrak{M}\ominus W\mathfrak{N})$, *and thus* $S^* \equiv H(\mathfrak{M}\ominus W\mathfrak{N})$. *If* (j, k) *is the* W-*deficiency index of* $\mathfrak{N}$, *then* (k, j) *is the deficiency index of* $H(\mathfrak{N})$. *Thus* $H(\mathfrak{N})$ *is maximal symmetric if and only if* $\mathfrak{N}$ *is maximal* W-*symmetric and* $H(\mathfrak{N})$ *is self-adjoint if and only if* $\mathfrak{N}$ *is hypermaximal* W-*symmetric.*

Let m *and* n *be the dimension numbers of* $\mathfrak{M}^+$ *and* $\mathfrak{M}^-$, *respectively. Then, if either* m *or* n *is zero,* H *is maximal symmetric and, if both* m *and* n *are zero,* H *is self-adjoint. If neither* m *nor* n *is zero, the class of all maximal symmetric extensions* S *of* H *has the cardinal number of the continuum. If* $m = n < \aleph_0$, *every maximal symmetric extension* S *of* H *is self-adjoint. If* m *exceeds* n (m *does not exceed* n), *every maximal symmetric extension* S *of* H *has the deficiency index* $(0, m - n)$ ($(n - m, 0)$). *If* $m = n = \aleph_0$ *and* p *is an arbitrary cardinal number on the range* $0 \le p \le \aleph_0$, *the class of all maximal symmetric extensions of* H *with the deficiency index* $(p, 0)$ ($(0, p)$) *has the cardinal number of the continuum.*

Excluding the trivial case $H \equiv H^*$, we consider first an arbitrary closed symmetric extension S of H. Then

$$\mathfrak{B}(S) \subset \mathfrak{B}^* = \mathfrak{D}(A), \quad A\mathfrak{B}(S) = A_1(\mathfrak{B}(S) \ominus \mathfrak{B}).$$

By Theorem 1.3, the latter manifold is W-symmetric in $\mathfrak{M}$. That it is closed is a consequence of the fact that A_1^{-1} exists and is bounded while $\mathfrak{B}(S) \ominus \mathfrak{B}$ is closed. On the other hand, if $\mathfrak{N}$ is an arbitrary closed linear W-symmetric manifold in $\mathfrak{M}$, $A_1^{-1}\mathfrak{N}$ is, by a similar argument, a closed linear manifold in $\mathfrak{B}^* \ominus \mathfrak{B} = \mathfrak{B}^+ + \mathfrak{B}^-$. Thus $A_1^{-1}\mathfrak{N} + \mathfrak{B}$ is closed; since, by Theorem 1.5, it is the graph of a linear symmetric extension S of H, S is closed.

Consider now an arbitrary element f of the domain of S^*. Then, since $WA\mathfrak{B}(S) = W\mathfrak{N}$, where S and $\mathfrak{N}$ correspond, Af belongs to $\mathfrak{M}\ominus W\mathfrak{N}$. On the other hand, since by Theorem 3.2 the range of A is $\mathfrak{M}$, we can determine a solution f in $\mathfrak{B}^*$ of the equation $Af = h$, for every h in $\mathfrak{M}\ominus W\mathfrak{N}$, and the solution f obviously belongs to the domain of S^*. Thus $A\mathfrak{B}(S^*) = \mathfrak{M}\ominus W\mathfrak{N}$, or $S^* \equiv H(\mathfrak{M}\ominus W\mathfrak{N})$.

Now let V be the isometric transformation with domain in $\mathfrak{M}^+$ and range in $\mathfrak{M}^-$ such that $\mathfrak{N} = \mathfrak{R}(I - V)$. Let $\mathfrak{M}_0 = (\mathfrak{M}^+ \ominus \mathfrak{D}(V)) + (\mathfrak{M}^- \ominus \mathfrak{R}(V))$. Let D be the transformation with domain $\mathfrak{B}(S^*)$ which takes f into $E_{\mathfrak{M}_0}Af$, where $E_{\mathfrak{M}_0}$ has domain $\mathfrak{M}$. Let W_0 be the contraction of W with domain $\mathfrak{M}_0$. We shall show that D is a reduction operator for S^*. Invoking Theorem 3.1, we can prove this by showing that the contraction D_1 of D with domain $\mathfrak{B}(S^*) \ominus \mathfrak{B}(S)$ is such that $D_1^* \equiv QD_1^{-1}W_0$, and this identity is readily shown to be equivalent to $D_1^{-1} \equiv QD_1^*W_0$. We shall prove the latter.

We observe first that since D_1 is bounded and has domain $\mathfrak{B}(S^*) \ominus \mathfrak{B}(S)$, D_1^* exists and has domain $\mathfrak{M}_0$. Furthermore, since $A\mathfrak{B}(S^*) = \mathfrak{M}\ominus W\mathfrak{N}$ and $A\mathfrak{B}(S) = \mathfrak{N}$, we see that

$$D_1(\mathfrak{B}(S^*) \ominus \mathfrak{B}(S)) = \mathfrak{M} \ominus (W\mathfrak{N} + \mathfrak{N}) = \mathfrak{M}_0.$$

And, as $S \equiv H(\mathfrak{N})$, if f is in $\mathfrak{B}(S^*)$ and $D\{f, S^*f\} = 0$, then $A\{f, S^*f\}$ is in $\mathfrak{N}$ and f is in $\mathfrak{D}(S)$. Consequently, we conclude that D_1^{-1}, as well as D_1^*, exists and has domain $\mathfrak{M}_0$. Therefore, to establish the identity $D_1^{-1} = QD_1^*W_0$, we need only show that $D_1^{-1}h = QD_1^*W_0h$ for an arbitrary element h of $\mathfrak{M}_0$.

Let h be such an element, and let $\{f, S^*f\} = D_1^{-1}h$. Then $WA\{f, S^*f\} = W_0h + Wk$, where k is in $\mathfrak{N}$. Thus, for all $\{g, S^*g\}$ in $\mathfrak{B}(S^*) \ominus \mathfrak{B}(S)$, we obtain

$$(g, S^*f) - (S^*g, f) + (Ag, W_0h + Wk) = 0.$$

But $Ag = D_1g + r$, where r is in $\mathfrak{N}$. Thus, because $(r, W_0h + Wk) = 0$, and $(Ag, Wk) = 0$, we have

$$(g, S^*f) - (S^*g, f) + (D_1g, W_0h) = 0$$

for all $\{g, S^*g\}$ in $\mathfrak{B}(S^*) \ominus \mathfrak{B}(S)$. Consequently, since by Theorem 2.8 $\{S^*f, -f\}$ is in $\mathfrak{B}(S^*) \ominus \mathfrak{B}$, W_0h is in the domain of D_1^* and $Q\{f, S^*f\} = -D_1^*W_0h$. Therefore, since $Q^{-1} \equiv -Q$, we have $QD_1^*W_0h = D_1^{-1}h$.

We have thus proved that D is a reduction operator for S^*. Furthermore, the characteristic manifolds of W_0 for the characteristic values $+i$ and $-i$ are evidently $\mathfrak{M}^+ \ominus \mathfrak{D}(V)$ and $\mathfrak{M}^- \ominus \mathfrak{R}(V)$, respectively; and by definition, the dimension numbers of these manifolds are j and k, respectively, where (j, k) is the W-deficiency index of V. Hence, by Theorem 3.7, (k, j) is the deficiency index of $S \equiv H(\mathfrak{N})$. From this it follows at once that $H(\mathfrak{N})$ is maximal symmetric if and only if $\mathfrak{N}$ is maximal W-symmetric, and that $H(\mathfrak{N})$ is self-adjoint if and only if $\mathfrak{N}$ is hypermaximal W-symmetric.

The assertions of the second paragraph of the theorem follow at once from Theorem 3.7 and known results; we state them here because they also follow at once from the assertions of the first paragraph and Theorem 2.5.

We shall find it convenient later to have the following simple facts stated precisely:

THEOREM 4.2. *There is a one-to-one correspondence between the class of all linear Q-symmetric manifolds $\mathfrak{N}$ in $\mathfrak{B}^+ + \mathfrak{B}^-$ and the class of all linear symmetric extensions S of H; S and $\mathfrak{N}$ correspond if and only if $\mathfrak{B}(S) = \mathfrak{B} + \mathfrak{N}$ and, when S and $\mathfrak{N}$ correspond,*

$$\mathfrak{B}(S^*) = \mathfrak{B} + ((\mathfrak{B}^+ + \mathfrak{B}^- \ominus Q\overline{\mathfrak{N}}).$$

S is closed if and only if $\mathfrak{N}$ is a closed linear manifold and, when S is closed, the deficiency index of S is the Q-deficiency index of $\mathfrak{N}$.

In so far as it applies to closed extensions S of H and closed manifolds $\mathfrak{N}$,

Theorem 4.2 can be deduced immediately from Theorem 4.1 by setting A equal to the reduction operator described in Theorem 2.9, the concluding assertion concerning the deficiency index of S requiring the additional observation that the Q-deficiency index of $\mathfrak{N}$ is (n, m), where (m, n) is its Q^{-1}-deficiency index. That this result can be extended to the more general one stated in Theorem 4.2 is readily verified, and the details may be left to the reader.

Theorems 2.7 and 4.2 contain in essence the basis of the theory of von Neumann in which symmetric extensions of a symmetric transformation H are determined by means of isometric extensions of its Cayley transform $(H-iI)(H+iI)^{-1}$; to perceive this one has only to note that every isometric transformation V from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ determines a unique isometric transformation X from $\mathfrak{D}^+$ to $\mathfrak{D}^-$, and conversely, according to the equation $V\{f^+, if^+\} = \{Xf^+, -iXf^+\}$.

3. **Unbounded reduction operators; preliminary questions.** For the case that A is bounded, we have in the preceding section characterized by means of boundary conditions not only all maximal symmetric extensions of H, but all closed symmetric extensions of H and their adjoints. In dealing with the case that A is unbounded, however, it is not feasible to take in so much territory. Instead, we restrict ourselves almost entirely to the twofold problem stated in Chapter I, §3.

In view of the fact that an unbounded reduction operator is not defined throughout the graph of H^*, it is natural to inquire first whether or not H has any maximal symmetric extensions S such that $\mathfrak{B}(S)$ belongs to $\mathfrak{D}(A)$, and whether or not it has any self-adjoint extensions S with that property. More generally we may ask if, given an arbitrary cardinal number p on the range $0 \leq p \leq \aleph_0$, there exist symmetric extensions S of H with deficiency index $(0, p)$ $((p, 0))$ such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$. For the discussion of this question, the classification of reduction operators into types I and II described in §4 of Chapter III is fundamental.

As we have already noted, a bounded reduction operator is of type II if and only if H has the deficiency index $(\aleph_0, \aleph_0)$. Thus, invoking the second paragraph of Theorem 4.1, we can say that a bounded reduction operator A for H^* has in its domain the graph of a maximal symmetric extension of H, with arbitrary preassigned deficiency index $(0, p)$ $((p, 0))$, if and only if A is of type II; and, taking account of Theorem 3.15, we may add that if A is of type I, every maximal symmetric S of H with $\mathfrak{B}(S) \subset \mathfrak{D}(A)$, has for its deficiency index the characteristic index of A. We shall now show that these assertions are valid even when A is unbounded.

THEOREM 4.3. *Let A be of type* II, *bounded or unbounded, and let p be an arbitrary cardinal number on the range $0 \leq p \leq \aleph_0$. Then the class of all maximal symmetric extensions S of H with the deficiency index $(0, p)$ $((p, 0))$ such that $\mathfrak{B}(S)$ belongs to $\mathfrak{D}(A)$ has the cardinal number $\mathfrak{c}$ of the continuum and coincides with the class of all maximal symmetric extensions S of H with deficiency index $(p, 0)$ $((0, p))$ if and only if A is bounded.*

As we have just observed, if A is bounded, H has the deficiency index $(\aleph_0, \aleph_0)$ and, as we know, $\mathfrak{D}(A) = \mathfrak{B}(H^*)$; thus every maximal symmetric extension S of H has its graph in $\mathfrak{D}(A)$, and the subclass of extensions S with deficiency index $(0, p)$ $((p, 0))$ has the cardinal number of the continuum.

Turning to the case that A is unbounded, we note first that the class of all symmetric extensions S of H, with deficiency index $(0, p)$ $((p, 0))$ has the cardinal number $\mathfrak{c}$. Hence the cardinal number of the subclass whose members S satisfy the condition $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ cannot be greater than $\mathfrak{c}$, so that we need only show that it is at least as great. Moreover, to establish the latter it is sufficient to exhibit a symmetric extension T of H with deficiency index $(\aleph_0, \aleph_0)$ such that $\mathfrak{B}(T^*) \subset \mathfrak{D}(A)$; for the class of all symmetric extensions S of T, with deficiency index $(0, p)$ $((p, 0))$ has the cardinal number $\mathfrak{c}$, and, if S is such a transformation, $\mathfrak{B}(S) \subset \mathfrak{B}(T^*) \subset \mathfrak{D}(A)$.

We now proceed to construct the transformation T. Again we denote by $\mathfrak{N}$ the range of $E_A(1-0)$ and by $\mathfrak{P}$ the manifold on which $(A_1^*A_1)^{1/2}$ is equal to I. Then, if $\mathfrak{P} \cdot \mathfrak{B}^+$ and $\mathfrak{P} \cdot \mathfrak{B}^-$ are both Hilbert spaces, the extension T whose graph is $\mathfrak{B} + \mathfrak{N}$ is symmetric and has the deficiency index $(\aleph_0, \aleph_0)$ by Theorem 4.2, since $\mathfrak{N}$ is Q-symmetric and has the Q-deficiency index $(\aleph_0, \aleph_0)$; furthermore $\mathfrak{B}(T^*) \subset \mathfrak{D}(A)$.

Now let us assume that at least one of the manifolds $\mathfrak{P} \cdot \mathfrak{B}^+$ and $\mathfrak{P} \cdot \mathfrak{B}^-$ is a unitary space. Then, according to Theorem 3.13, the transformation induced in $\mathfrak{N}$ by $(A_1^*A_1)^{1/2}$ is not totally continuous and thus, by Lemma 3.1, has in its range a closed linear manifold $\mathfrak{N}_1$ with the dimension number $\aleph_0$. Furthermore, $\mathfrak{N}_1$ is clearly Q-symmetric because $\mathfrak{N}$ is, and $Q\mathfrak{N}_1$ is in $\mathfrak{D}(A)$ by Theorems 3.5 and 3.6. Thus $\mathfrak{N}_1 + Q\mathfrak{N}_1$ is in $\mathfrak{D}(A)$. But, applying Theorem 2.2, we see that $\mathfrak{N}_1 + Q\mathfrak{N}_1 = \mathfrak{D}(X) + \mathfrak{R}(X)$, where $\mathfrak{N}_1 = \mathfrak{R}(I - X)$ and X is a closed isometric transformation with domain in $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$. Thus $\mathfrak{D}(X) \subset \mathfrak{D}(A)$ and $\mathfrak{R}(X) \subset \mathfrak{D}(A)$. Moreover, since $\mathfrak{N}_1$ is a Hilbert space, $\mathfrak{D}(X)$ and $\mathfrak{R}(X)$ must be Hilbert spaces. Now let T be the transformation in $\mathfrak{H}$ whose graph is $\mathfrak{B} + (\mathfrak{N} \ominus \mathfrak{N}_1)$. Then by Theorem 4.2, since $\mathfrak{N} \ominus \mathfrak{N}_1$ is closed linear Q-symmetric, T is a closed linear symmetric extension of H. Furthermore,

$$\mathfrak{B}(T^*) = \mathfrak{B} + (\mathfrak{N} \ominus \mathfrak{N}_1) + \mathfrak{D}(X) + \mathfrak{R}(X) + \mathfrak{P},$$

and therefore T has the deficiency index $(\aleph_0, \aleph_0)$ while $\mathfrak{B}(T^*) \subset \mathfrak{D}(A)$, as we wished to show.

To complete the proof of the theorem it is necessary only to show that when A is unbounded, there exist maximal symmetric extensions S of H, with deficiency index $(0, p)$ $((p, 0))$ such that $\mathfrak{B}(S) \not\subset \mathfrak{D}(A)$. To construct such an extension S, we start with a symmetric extension S_1 of H with the desired deficiency index but such that $\mathfrak{B}(S_1) \subset \mathfrak{D}(A)$ and denote by X_1 the isometric transformation with domain in $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$ such that $\mathfrak{B}(S_1) \ominus \mathfrak{B}$ $= \mathfrak{R}(I - X_1)$. Then either $\mathfrak{D}(X_1) = \mathfrak{B}^+$ or $\mathfrak{R}(X_1) = \mathfrak{B}^-$. Furthermore, since $\mathfrak{R}(I - X_1) \subset \mathfrak{D}(A)$, $\{f^+, if^+\}$ in $\mathfrak{D}(X_1)$ belongs to $\mathfrak{B}_A^+ = \mathfrak{B}^+ \cdot \mathfrak{D}(A)$ if and only if $X_1 \{f^+, if^+\}$ belongs to $\mathfrak{B}_A^- = \mathfrak{B}^- \cdot \mathfrak{D}(A)$. Thus we can determine an element $\{f^+, if^+\}$ of $\mathfrak{D}(X_1)$ such that neither $\{f^+, if^+\}$ nor $X_1\{f^+, if^+\}$ belongs to $\mathfrak{D}(A)$. Therefore, since $(I - X_1)\{f^+, if^+\}$ belongs to $\mathfrak{D}(A)$, $(I + X_1)\{f^+, if^+\}$ is not contained in $\mathfrak{D}(A)$. Consequently, if X is the transformation with domain $\mathfrak{D}(X_1)$ which is equal to $-X_1$ on the manifold determined by $\{f^+, if^+\}$ and to X_1 on the manifold in $\mathfrak{D}(X_1)$ perpendicular to $\{f^+, if^+\}$, then $\mathfrak{B} + \mathfrak{R}(I - X)$ is the graph of a symmetric extension S of H with the same deficiency index as S_1 and $\mathfrak{B}(S) \not\subset \mathfrak{D}(A)$.

The proof of the theorem is thus complete.·

We turn now to unbounded operators of type I, beginning with a necessary lemma.

LEMMA 4.1. *Let $\mathfrak{L}$ be a Hilbert space and V a closed isometric transformation in $\mathfrak{L}$, with deficiency index (j, k), $j \neq k$. Then the range of $I - V$ contains a Hilbert space—that is, a closed linear manifold with the dimension number $\aleph_0$.*

We denote by $\mathfrak{L}_1$ the manifold on which $V = I$. Then it is readily shown that $\mathfrak{R}(I - V) \subseteq \mathfrak{L} \ominus \mathfrak{L}_1$. Now let V_1 be the transformation with domain $(\mathfrak{L} \ominus \mathfrak{L}_1) \cdot \mathfrak{D}(V)$ which is equal on its domain to V. Then $\mathfrak{R}(V_1) \subset \mathfrak{L} \ominus \mathfrak{L}_1$. Thus, since $\mathfrak{L}_1 \subset \mathfrak{D}(V)$ and $\mathfrak{L}_1 \subset \mathfrak{R}(V)$, V_1 is an isometric transformation in $\mathfrak{L} \ominus \mathfrak{L}_1$ with the deficiency index (j, k). Moreover, since $j \neq k$, $\mathfrak{L} \ominus \mathfrak{L}_1$ is clearly a Hilbert space.

We now determine in $\mathfrak{L} \ominus \mathfrak{L}_1$ a maximal symmetric extension V_2 of V_1 such that $(I - V_2)^{-1}$ exists; for this construction it is necessary only to choose a maximal isometric transformation V_3 from $(\mathfrak{L} \ominus \mathfrak{L}_1) \ominus \mathfrak{D}(V_1)$ to $(\mathfrak{L} \ominus \mathfrak{L}_1) \ominus \mathfrak{R}(V_1)$ such that $V_3 \neq I$ at any point of its domain and then to define $V_2 = V_1$ on $\mathfrak{D}(V_1)$, $V_2 = V_3$ on $\mathfrak{D}(V_3)$. Evidently the deficiency index of V_2 in $\mathfrak{L} \ominus \mathfrak{L}_1$ is either $(j - k, 0)$ or $(0, k - j)$, according as j does or does not exceed k.

Next we observe that $I - V_2$ has range dense in $\mathfrak{L} \ominus \mathfrak{L}_1$. For, if $(f - V_2 f, g) = 0$ for all f in $\mathfrak{D}(V_2)$ and some g in $\mathfrak{L} \ominus \mathfrak{L}_1$, we have $(f, g) = (V_2 f, g)$ and this is

readily shown to imply $g = V_2 g$, which in turn implies $g = 0$. Consequently $S_2 \equiv (I + V_2)(I - V_2)^{-1}$ is a maximal symmetric transformation in $\mathfrak{L} \ominus \mathfrak{L}_1$ and

$$S_2 \supseteq S_1 \equiv (I + V_1)(I - V_1)^{-1}.\dagger$$

Furthermore, applying Theorems 2.7 and 2.3, we have $\mathfrak{B}(S_2) \ominus \mathfrak{B}(S_1) = \mathfrak{B}(S_3)$, where $S_3 \equiv (I + V_3)(I \div V_3)^{-1}$ and V_3 has the same meaning as above. More-over, $\mathfrak{B}(S_3)$ is a unitary space, because its dimension number is the minimum of j and k and, since $j \neq k$, the minimum of the two is finite.

Now let us suppose that $\mathfrak{D}(S_2)$ contains a Hilbert space $\mathfrak{L}_2$ and denote by S_4 the contraction of S_2, with domain $\mathfrak{L}_2$. Then S_4 is evidently closed and $\mathfrak{B}(S_4)$ is thus a closed linear manifold in $\mathfrak{B}(S_2)$. Let T be the transformation with domain $\mathfrak{B}(S_4)$ and range in $\mathfrak{B}(S_3)$ which takes each element of its do-main into the projection of that element on $\mathfrak{B}(S_3)$, let $\mathfrak{N}$ be the manifold of zeros of T, and let $\mathfrak{N}_1 = \mathfrak{B}(S_4) \ominus \mathfrak{N}$. Then the dimension number of $\mathfrak{N}_1$ is clearly not greater than that of $\mathfrak{B}(S_3)$. Thus, since $\mathfrak{B}(S_4)$ is a Hilbert space and $\mathfrak{B}(S_3)$ a unitary space, $\mathfrak{N}$ must be a Hilbert space. Moreover, $\mathfrak{N} = \mathfrak{B}(S_4) \cdot \mathfrak{B}(S_1)$ and is evidently the graph of a closed linear transforma-tion S_5, $S_5 \subseteq S_1$, $S_5 \subseteq S_4$. But S_4, having domain the closed space $\mathfrak{L}_2$ and being closed itself, is therefore bounded.‡ Hence S_5 is bounded; therefore, since S_5 is also closed, $\mathfrak{D}(S_5)$ is a closed linear manifold. Furthermore, $\mathfrak{D}(S_5)$ has the dimension number $\aleph_0$, because $\mathfrak{B}(S_5)$ has that dimension number, and

$$\mathfrak{D}(S_5) \subset \mathfrak{D}(S_1) = \mathfrak{R}(I - V_1) = \mathfrak{R}(I - V).$$

In consequence of the result just obtained, it is necessary for the comple-tion of the proof only to show that the domain of the maximal symmetric transformation S_2 contains a Hilbert space $\mathfrak{L}_2$. To prove this we first recall that since S_2 is maximal symmetric and not self-adjoint, there is a Hil-bert space $\mathfrak{L}_3$ which reduces S_2 and in which S_2 induces an elementary sym-metric transformation.§ Therefore, by definition, in $\mathfrak{L}_3$, $S_2 = (I + X)(I - X)^{-1}$, where X is defined in $\mathfrak{L}_3$, in terms of some complete orthonormal set $\{\phi_n\}$, $n = 1, 2, 3, \cdots$, by the equations $X\phi_n = \phi_{n+1}$, $n = 1, 2, 3, \cdots$, or by the equations $X\phi_n = \phi_{n-1}$, $n = 2, 3, \cdots$.‖ From this fact it is evident that the orthonormal set $\{(\phi_n - \phi_{n-1})/2\}$, $n = 2, 3, \cdots$, determines a Hilbert space $\mathfrak{L}_2$ which belongs to $\mathfrak{R}(I - X)$ and thus to $\mathfrak{D}(S_2)$. Consequently the lemma is true as stated.

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† Stone, Theorems 9.1 and 9.2.
‡ Murray, Theorem 1.25.
§ Stone, Theorem 9.10.
‖ Stone, Theorem 9.9 and Definition 9.6.

THEOREM 4.4. *Let A be a reduction operator of type* I. *Then, if H is not maximal symmetric, the class of all maximal symmetric extensions S of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ has the cardinal number $\mathfrak{c}$ of the continuum and coincides with the class of all maximal symmetric extensions S of H if and only if A is bounded. If S is a maximal symmetric extension of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$, then the deficiency index of S is the characteristic index of A. If A is unbounded, there exist maximal symmetric extensions S of H with this deficiency index such that $\mathfrak{B}(S) \not\subset \mathfrak{D}(A)$.*

If A is bounded, $\mathfrak{D}(A) = \mathfrak{B}(H^{*})$, and every maximal symmetric extension S of H has its graph in $\mathfrak{D}(A)$; furthermore, if H is not maximal, the cardinal number of the class of all such extensions S is the cardinal number $\mathfrak{c}$ of the continuum, by Theorem 4.1. That every such extension has for its deficiency index the characteristic index of A follows from Theorems 3.15 and 4.1.

Now let A be unbounded. Invoking an argument like that used in the proof of Theorem 4.3, we can prove that the class defined in the theorem has the cardinal number $\mathfrak{c}$ by exhibiting a subclass with that cardinal number. Moreover, for this purpose it is clearly sufficient to construct a closed symmetric extension S_1 of H, which is not maximal and which satisfies the condition $\mathfrak{B}(S_1^{*}) \subset \mathfrak{D}(A)$.

To construct S_1, we start with the transformation S_2 whose graph is $\mathfrak{N} + \mathfrak{B}$ where $\mathfrak{N}$ is the range of $E_A(1-0)$. Then by Theorem 4.2, if $\mathfrak{P}$ is the manifold on which $(A_1^{*}A_1)^{1/2} = I$, S_2 is a closed symmetric extension of H with the deficiency index (j, k), where j and k are the dimension numbers of $\mathfrak{P} \cdot \mathfrak{B}^{+}$ and $\mathfrak{P} \cdot \mathfrak{B}^{-}$, respectively. Furthermore, since $\mathfrak{B}_A^{+}$ contains $\mathfrak{P} \cdot \mathfrak{B}^{+}$ and is dense in $\mathfrak{B}^{+}$, $(\mathfrak{B}^{+} \ominus \mathfrak{B}^{+} \cdot \mathfrak{P}) \cdot \mathfrak{B}_A^{+}$ contains an element $\{f^{+}, if^{+}\} \neq 0$. Now let X_2 be the isometric transformation with domain $(\mathfrak{B}^{+} \ominus \mathfrak{B}^{+} \cdot \mathfrak{P})$ and range $(\mathfrak{B}^{-} \ominus \mathfrak{B}^{-} \cdot \mathfrak{P})$ such that $\mathfrak{N} = \mathfrak{R}(I - X_2)$, and let X_1 be the contraction of X_2 whose domain is the set of elements of $\mathfrak{D}(X_2)$ orthogonal to $\{f^{+}, if^{+}\}$. Then by Theorem 4.2, $\mathfrak{B} + \mathfrak{R}(I - X_1)$ is the graph of a symmetric extension S_1 of S and

$$\mathfrak{B}(S_1^{*}) = \mathfrak{B} + \mathfrak{R}(I - X_1) + \mathfrak{B}^{+} \cdot \mathfrak{P} + \mathfrak{B}^{-} \cdot \mathfrak{P} + \mathfrak{B}_i^{+} + \mathfrak{B}_i^{-},$$

where $\mathfrak{B}_i^{+}$ is the linear manifold determined by $\{f^{+}, if^{+}\}$, $\mathfrak{B}_i^{-}$ the linear manifold determined by $X_2\{f^{+}, if^{+}\}$. Moreover, since $\{f^{+}, if^{+}\}$ and $(I - X)\{f^{+}, if^{+}\}$ belong to $\mathfrak{D}(A)$, $X_2\{f^{+}, if^{+}\}$ does also. Thus $\mathfrak{B}(S_1^{*}) \subset \mathfrak{D}(A)$. Therefore, since S_1 clearly has the deficiency index $(j+1, k+1)$, the class of maximal symmetric extensions S of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ has the cardinal number $\mathfrak{c}$.

To prove that every such extension has the deficiency index stated in the theorem we note first that every maximal symmetric extension S, of the transformation S_1 just constructed, has for its deficiency index the character-

istic index of A (that is, either $(j-k, 0)$ or $(0, k-j)$ according as j does or does not exceed k). Let S be such an extension, and let $\mathfrak{B}(S) \ominus \mathfrak{B} = \mathfrak{R}(I-X)$, where X is the closed isometric transformation with domain in $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$, which exists in accordance with Theorems 4.2 and 2.2. Let T be another maximal symmetric extension of H such that $\mathfrak{B}(T) \subset \mathfrak{D}(A)$, and let $\mathfrak{B}(T) \ominus \mathfrak{B} = \mathfrak{R}(I-Y)$ where Y is closed isometric with domain in $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$. Then, since both $\mathfrak{R}(I-X)$ and $\mathfrak{R}(I-Y)$ belong to $\mathfrak{D}(A)$, $\mathfrak{R}(X-Y)$ belongs to $\mathfrak{D}(A)$. Moreover,

$$\mathfrak{R}(I - X) = \mathfrak{R}((I - X)X^{-1}) = \mathfrak{R}(I - X^{-1})$$

and, similarly,

$$\mathfrak{R}(I - Y) = \mathfrak{R}(I - Y^{-1}),$$

so that $\mathfrak{R}(X^{-1}-Y^{-1})$ belongs to $\mathfrak{D}(A)$ also. Thus $\mathfrak{R}(X-Y)$ is in $\mathfrak{B}_A^-$ and $\mathfrak{R}(X^{-1}-Y^{-1})$ is in $\mathfrak{B}_A^+$. Furthermore,

$$\mathfrak{R}(X - Y) = \mathfrak{R}((X - Y)Y^{-1}) = \mathfrak{R}(I - XY^{-1})$$

and

$$\mathfrak{R}(X^{-1} - Y^{-1}) = \mathfrak{R}((X^{-1} - Y^{-1})X) = \mathfrak{R}(I - Y^{-1}X).$$

Now let us suppose that the deficiency index of S is $(0, p)$ and that of T is $(q, 0)$. Then XY^{-1} is a maximal isometric transformation in $\mathfrak{B}^-$ with deficiency index $(0, p+q)$ and $Y^{-1}X$ is a maximal symmetric transformation in $\mathfrak{B}^+$ with the same deficiency index. Hence, since $\mathfrak{R}(I-XY^{-1}) \subseteq \mathfrak{B}_A^-$ and $\mathfrak{R}(I-Y^{-1}X) \subseteq \mathfrak{B}_A^+$, it follows, from Lemma 4.1 and the hypothesis that A is of type I, that $p=q=0$. Moreover, by an entirely similar argument, the assumption that S has the deficiency index $(p, 0)$ and T the deficiency index $(0, q)$ leads to the same conclusion.

Next let us assume that S has the deficiency index $(p, 0)$, T the deficiency index $(q, 0)$; then $Y^{-1}X$ has the deficiency index (p, q). Let us assume $p \neq q$. Then, by Lemma 4.1, since $\mathfrak{B}_A^+ \supseteq \mathfrak{R}(I-Y^{-1}X)$, $\mathfrak{B}_A^+$ contains a Hilbert space $\mathfrak{B}_2^+$. Furthermore, since either p or q is finite, either $\mathfrak{B}^+ \ominus \mathfrak{D}(X)$ or $\mathfrak{B}^+ \ominus \mathfrak{D}(Y)$ is a unitary space. Therefore, by an argument like one used in the proof of Lemma 4.1, either $\mathfrak{D}(X) \cdot \mathfrak{B}_2^+$ or $\mathfrak{D}(Y) \cdot \mathfrak{B}_2^+$ is a Hilbert space $\mathfrak{B}_3^+$. Moreover, since $\mathfrak{R}(I-X)$, $\mathfrak{R}(I-Y)$, and $\mathfrak{B}_2^+$ belong to $\mathfrak{D}(A)$, either X or Y takes $\mathfrak{B}_3^+$ into a Hilbert space in $\mathfrak{B}_A^-$. But this is evidently a contradiction of the hypothesis that A is of type I, and we conclude in consequence that the assumption $p \neq q$ is untenable.

Since the case that S and T have deficiency indices $(0, p)$ and $(0, q)$, respectively, can evidently be handled in a manner entirely similar to the above, we conclude that T has the same deficiency index as S.

To complete the proof, it is now only necessary to show that when A is unbounded there exists a maximal symmetric extension S of H, whose deficiency index is the characteristic index of A and whose graph is not in $\mathfrak{D}(A)$. A method for the proof of this, however, has already been used in proving a similar assertion in Theorem 4.2; we leave the details here to the reader.

We now prove a theorem which establishes the significance of the second part of our fundamental problem, stated in Chapter I, §3.

THEOREM 4.5. *Let A be unbounded, and let p be an arbitrary cardinal number on the range $0 \leqq p \leqq \aleph_0$. Let S be an arbitrary maximal symmetric extension of H. Then $\mathfrak{B}(S) \cdot \mathfrak{D}(A)$ is the graph of a symmetric extension S_A of H. The class of all maximal symmetric extensions S of H, with the deficiency index $(0, p)$ $((p, 0))$, such that $S_A \not\equiv S$, $\tilde{S}_A \equiv S$, has the cardinal number $\mathfrak{c}$ of the continuum.*

The first assertion of the theorem is obvious. To prove the second, we observe first that, since the linear manifold $\mathfrak{B}_A{}^+ = \mathfrak{D}(A) \cdot \mathfrak{B}^+$ is dense in $\mathfrak{B}^+$ and the linear manifold $\mathfrak{B}_A{}^- = \mathfrak{D}(A) \cdot \mathfrak{B}^-$ is dense in $\mathfrak{B}^-$, we can choose in $\mathfrak{B}_A{}^+$ an orthonormal set $\{f_n^+, if_n^+\}$ complete in $\mathfrak{B}^+$ and in $\mathfrak{B}_A{}^-$ an orthonormal set $\{f_n^-, -if_n^-\}$ complete in $\mathfrak{B}^-$. Hence if V is the closed isometric transformation defined in terms of the two sets by the equation

$$V\{f_n^+, if_n^+\} = \{f_{n+p}^-, -if_{n+p}^-\}, \qquad (V\{f_{n+p}^+, if_{n+p}^+\} = \{f_n^-, -if_n^-\}),$$

$\mathfrak{B} + \mathfrak{R}(I-V)$ is clearly the graph of a maximal symmetric extension S of H, with deficiency index $(0, p)$ $((p, 0))$, such that the transformation S_A defined in the theorem has S for its closure. Moreover, the class of all transformations S which can be constructed in this way evidently has the cardinal number $\mathfrak{c}$. Finally, if $S \equiv S_A$, the transformation T whose graph is $\mathfrak{B} + \mathfrak{R}(I+V)$ obviously has the same deficiency index as S and satisfies the conditions $T_A \not\equiv T$, $\tilde{T}_A \equiv T$. Thus, again taking account of the fact that the class of all maximal symmetric extensions of H with deficiency index $(0, p)$ $((p, 0))$ has the cardinal number $\mathfrak{c}$, we conclude that the class defined in the theorem has the same cardinal number.

It is to be emphasized that when A is unbounded, the maximal symmetric extensions of H described in Theorems 4.3, 4.4, and 4.5, do not exhaust the class of all maximal symmetric extensions of H. We now indicate briefly the wide range of other possibilities.

We begin by considering an arbitrary maximal symmetric extension S_1 of H such that $\mathfrak{B}(S_1) \subset \mathfrak{D}(A)$. For simplicity, let us assume that the deficiency index of S_1 is $(0, p)$. Then $\mathfrak{B}(S_1) \ominus \mathfrak{B} = \mathfrak{R}(I-X)$, where X is isometric with domain $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$. Now let T be any self-adjoint transformation in $\mathfrak{B}^+$, and let $V \equiv (T-iI)(T+iI)^{-1}$. Then V is unitary in $\mathfrak{B}^+$, and $Y \equiv XV$

is isometric with domain $\mathfrak{B}^+$ and range in $\mathfrak{B}^-$. Thus, by Theorems 2.2 and 4.2, $\mathfrak{B} + \mathfrak{R}(I - Y) = \mathfrak{B} + \mathfrak{R}(I - Y^{-1})$ is the graph of a maximal symmetric extension S of H. Furthermore, for every h in $\mathfrak{R}(Y) = \mathfrak{R}(X)$, we have

$$h - Y^{-1}h = (h - X^{-1}h) + (X^{-1}h - V^{-1}X^{-1}h).$$

Consequently since $\mathfrak{R}(I - X^{-1})$ is in $\mathfrak{D}(A)$ by hypothesis, $h - Y^{-1}h$ is in $\mathfrak{D}(A)$ if and only if $(X^{-1}h - V^{-1}X^{-1}h)$ is in $\mathfrak{D}(A)$ and thus in $\mathfrak{B}_A{}^+ = \mathfrak{D}(A) \cdot \mathfrak{B}^+$. But $\mathfrak{R}(X^{-1}) = \mathfrak{B}^+$ and $\mathfrak{R}(I - V^{-1}) = \mathfrak{D}(T)$. Therefore $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ if and only if $\mathfrak{D}(T) \subseteq \mathfrak{B}_A{}^+$ and $\mathfrak{B}(S) \cdot \mathfrak{D}(A)$ is dense in $\mathfrak{B}(S)$ if and only if T is the closure of a transformation T_1 with domain in $\mathfrak{B}_A{}^+$.

Evidently, when A is unbounded T can be chosen in a wide variety of ways so that neither of these conditions is satisfied; in particular, we can proceed as follows. According to Theorem 3.10, $\mathfrak{B}_A{}^+$ is the range of a closed transformation M with domain $\mathfrak{M}^-$, such that M^{-1} exists. Hence $T_0 = (M^{-1}(M^*)^{-1})^{1/2}$ is a self-adjoint transformation with domain $\mathfrak{B}_A{}^+$. Furthermore, if A is unbounded, $\mathfrak{B}_A{}^+ \neq \mathfrak{B}^+$ and T_0 is unbounded. Consequently there exists a unitary transformation U in $\mathfrak{B}^+$ such that $\mathfrak{D}(UT_0U^{-1}) \cdot \mathfrak{D}(T_0) = \mathfrak{O}$. Let $T \equiv UT_0U^{-1}$.[†] Then, if S is the maximal symmetric extension of H determined by T and the extension S_1 of H in the manner described immediately above, we clearly have $\mathfrak{B}(S) \cdot \mathfrak{D}(A) = \mathfrak{B}$. Thus, since an entirely similar procedure is possible if the transformation S_1 has the deficiency index $(p, 0)$, we can state the following theorem:

THEOREM 4.6. *If A is an unbounded reduction operator, there exist maximal symmetric extensions S of H such that $\mathfrak{B}(S)$ and $\mathfrak{D}(A)$ intersect only on the graph of H.*

Further light on the pathological aspects of the theory of unbounded reduction operators is provided by the following theorem which was communicated to the author by J. von Neumann: If H has the deficiency index $(\aleph_0, \aleph_0)$, there exist unbounded reduction operators A and C for H^* such that $\mathfrak{D}(A) \cdot \mathfrak{D}(C) = \mathfrak{B}$. In fact, it can be shown that the operator A can be an arbitrary unbounded reduction operator for H^* and C determined so that $\mathfrak{D}(A) \cdot \mathfrak{D}(C) = \mathfrak{B}$. Thus Theorem 4.6 can be obtained as a consequence of Theorems 4.3 and 4.4.

4. **Boundary conditions involving an arbitrary reduction operator.** We now return to the consideration of the maximal symmetric extensions of H described in Theorems 4.3, 4.4, and 4.5. According to Theorems 1.2 and 1.3, each such extension S, or a transformation S_A whose closure is S, can be de-

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[†] Von Neumann, Journal für die reine und angewandte Mathematik (Crelle), vol. 161 (1929), pp. 208–236; Theorem 17.

fined by means of a boundary condition of the kind described in Definition 1.2. In order to describe what special properties of the boundary condition are equivalent to the maximal property of the transformation S, it is convenient to vary slightly our previous notation.

DEFINITION 4.1. *Let $\mathfrak{N}$ be a linear W-symmetric manifold in $\mathfrak{M}$, and let V be the isometric transformation with domain in $\mathfrak{M}^+$ and range in $\mathfrak{M}^-$, such that $\mathfrak{N} = \mathfrak{R}(I - V)$ associated with $\mathfrak{N}$ by Theorem 2.2. Then $H(V)$ denotes the same transformation as $H(\mathfrak{N})$, where $H(\mathfrak{N})$ is the operator defined in Definition 1.2.*

THEOREM 4.7. *The transformation $H(V)$ of Definition 4.1 is a linear symmetric extension of H. If S is an arbitrary linear symmetric extension of H such that $\mathfrak{B}(S) \subseteq \mathfrak{D}(A)$, there exists an isometric transformation V with domain in $\mathfrak{M}^+$ and range in $\mathfrak{M}^-$ such that $S \equiv H(V)$.*

Theorem 4.7 is essentially only a restatement of Theorem 1.5.

We emphasize that $H(V_1) \equiv H(V_2)$ does not imply $V_1 \equiv V_2$. For example, V_1 can be the transformation whose domain contains only the element 0 of $\mathfrak{M}^+$, while V_2 has for its domain the linear manifold determined by an element h^+ of $\mathfrak{M}^+$ such that $h^+ - V_2 h^+$ is not in $\mathfrak{R}(A)$. We then have $H \equiv H(V_1) \equiv H(V_2)$.

THEOREM 4.8. *Let F be the transformation defined in Theorem 3.11, and let $\mathfrak{M}_A{}^+$ and $\mathfrak{M}_A{}^-$ have the same meanings as in Theorem 3.8. Then a necessary and sufficient condition that the transformation $H(V)$ of Definition 4.1 be maximal symmetric with deficiency index $(0, p)$ $((p, 0))$ is that $\mathfrak{R}(V - F) \supseteq \mathfrak{M}_A{}^-$ $(\mathfrak{R}(V^{-1} - F^*) \supseteq \mathfrak{M}_A{}^+)$. Thus $H(V)$ is self-adjoint if and only if $\mathfrak{R}(V - F) \supseteq \mathfrak{M}_A{}^-$ and $\mathfrak{R}(V^{-1} - F^*) \supseteq \mathfrak{M}_A{}^+$.*

If S is an arbitrary maximal symmetric extension of H and $\mathfrak{B}(S) \subseteq \mathfrak{D}(A)$, there exists one and only one closed isometric transformation V from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that the boundary condition $Af \, \varepsilon \, \mathfrak{R}(I - V)$ is nondegenerate and such that $S \equiv H(V)$.

Our initial step in the proof of the first portion of the theorem is purely formal, leading to an equation on which the proof is based. We consider an arbitrary element $\{f, H^*f, Af\}$ of $\mathfrak{B}(A)$ such that $Af \, \varepsilon \, \mathfrak{R}(I - V)$ and set

$$Af = h^+ - V h^+,$$

$$\{f, H(V)f\} = \{f^+, if^+\} - X\{f^+, if^+\} + \{f_0, Hf_0\},$$

X being the unique isometric transformation from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that $\mathfrak{R}(I - X) = \mathfrak{B}(S) \cdot (\mathfrak{B}^+ + \mathfrak{B}^-)$, whose existence is assured by Theorems 4.2 and 2.2. We then apply Theorem 3.9 to write

$$\{f, H(V)f, Af\} = \{f_-, H^*f_-, Af_-\} - \{f^-, -if^-, Af^-\} + \{f_0, Hf_0, 0\},$$

where the first component on the right is in $\mathfrak{B}_-(A)$, the second in $\mathfrak{B}^-(A)$. Recalling that Af_- is in $\mathfrak{M}^-$, and comparing the first and third equations above, we have

$$-Af^- = h^+ - Fh^+, \qquad Af_- = -(V - F)h^+.$$

Likewise, comparing the second and third, we obtain

$$\{f_-, H^*f_-\} = \{f^+, if^+\} - G\{f^+, if^+\}, \quad \{f^-, -if^-\} = (X - G)\{f^+, if^+\},$$

where G has the same meaning as in Theorem 3.11. Consequently,

(4.1) $A_1(I - G)\{f^+, if^+\} = -(V - F)h^+$

for all $\{f^+, if^+\}$ in $\mathfrak{D}(X)$ or, to put it differently, for all h^+ in $\mathfrak{D}(V)$ such that $h^+ - Vh^+ \,\varepsilon\, \mathfrak{R}(A)$.

Thus from (4.1) we have

(4.2) $A_1(I - G)\mathfrak{D}(X) \subseteq \mathfrak{R}(V - F).$

Furthermore A_1^{-1} and $(I-G)^{-1}$ exist; so we may write

(4.3) $\mathfrak{D}(X) \subseteq (I - G)^{-1}A_1^{-1}[\mathfrak{R}(V - F)\cdot\mathfrak{R}(A)].$

But, if $(V-F)h^+$ is in $\mathfrak{R}(A)$, then

$$(I - V)h^+ = (I - F)h^+ - (V - F)h^+$$

is also, since $(I-F)h^+$ is in $\mathfrak{R}(A)$ by definition of F. Therefore (4.3) becomes

(4.4) $\mathfrak{D}(X) = (I - G)^{-1}A_1^{-1}[\mathfrak{R}(V - F)\cdot\mathfrak{M}_A^-],$

where we have made use of the relations $\mathfrak{R}(V-F) \subseteq \mathfrak{M}^-$, $\mathfrak{M}_A^- = \mathfrak{M}^-\cdot\mathfrak{R}(A)$; in addition, we can evidently replace (4.2) by

(4.5) $A_1(I - G)\mathfrak{D}(X) = \mathfrak{R}(V - F)\cdot\mathfrak{M}_A^-.$

But, by Theorem 3.11, (3), $A_1(I-G)\mathfrak{B}^+ = \mathfrak{M}_A^-$, and besides, it is clear that $\mathfrak{B}^+ = (I-G)^{-1}A_1^{-1}\mathfrak{M}_A^-$. Consequently we conclude from equations (4.4) and (4.5) that $\mathfrak{D}(X) = \mathfrak{B}^+$ if and only if $\mathfrak{R}(V-F)\cdot\mathfrak{M}_A^- = \mathfrak{M}_A^-$; that is to say, if and only if $\mathfrak{R}(V-F) \supseteq \mathfrak{M}_A^-$. Moreover, $\mathfrak{D}(X) = \mathfrak{B}^+$ if and only if $\mathfrak{R}(I-X)$ is maximal Q-symmetric with Q-deficiency index $(0, p)$. Hence, by Theorem 4.2, $H(V)$ has deficiency index $(0, p)$ if and only if $\mathfrak{R}(V-F) \supseteq \mathfrak{M}_A^-$, as we wished to prove.

On the other hand, if we make use of the resolution $\{f, H(V)f, Af\} = \{f_+, H^*f_+, Af_+\} + \{f^+, if^+, Af^+\} + \{f_0, Hf_0, 0\}$, where the first component is in $\mathfrak{B}_+(A)$, and the second in $\mathfrak{B}^+(A)$, also provided in Theorem 3.9, an entirely similar argument yields the result that $H(V)$ has deficiency index $(p, 0)$ if and only if $\mathfrak{R}(V^{-1}-F^*) \supseteq \mathfrak{M}_A^+$. We leave the details here to the reader.

Since $H(V)$ is self-adjoint if and only if it has deficiency index $(0, 0)$, the concluding assertion of the first paragraph of the theorem follows at once.

To prove the proposition formulated in the second paragraph of the theorem, we consider an arbitrary maximal symmetric extension S of H, $\mathfrak{B}(S) \subset \mathfrak{D}(A)$, and denote by $\mathfrak{N}$ the closure of $A\mathfrak{B}(S)$. Then, if V is the isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $\mathfrak{N} = \mathfrak{R}(I - V)$, V is closed by Theorem 2.2, $S \equiv H(V)$, and the boundary condition $Af \, \varepsilon \, \mathfrak{R}(I - V)$ is obviously nondegenerate since $A\mathfrak{B}(S)$ has $\mathfrak{R}(I - V)$ for its closure. Now suppose that $S \equiv H(V_1)$ and that V_1 is closed. Then $\mathfrak{R}(I - V_1)$ is closed and, since $\mathfrak{R}(I - V_1) \supseteq A\mathfrak{B}(S)$, we have $\mathfrak{R}(I - V_1) \supseteq \mathfrak{R}(I - V)$. Hence, by Theorem 2.2, $V_1 \supseteq V$. But if $V_1 \supset V$, $\mathfrak{D}(V_1)$ contains an element $h \neq 0$ perpendicular to $\mathfrak{D}(V)$ and $h - V_1 h$ is thus perpendicular to $\mathfrak{R}(I - V)$. Furthermore there can be no element in $\mathfrak{R}(I - V_1) \cdot \mathfrak{R}(A)$ which is not in $\mathfrak{R}(I - V)$, since otherwise S would not be maximal. Therefore $V_1 \supset V$ implies that the condition $Af \, \varepsilon \, \mathfrak{R}(I - V_1)$ is degenerate, and the proof is complete.

That when A is unbounded there do exist degenerate boundary conditions defining maximal symmetric extensions of H will be proved in the next section.

It is now easy to prove, by arguments of the same tenor as those used to establish the first portion of the preceding theorem, the following statement:

THEOREM 4.9. *A necessary and sufficient condition that the transformation $H(V)$ of Definition 4.1 have a maximal symmetric closure with deficiency index $(0, p)$ $((p, 0))$ is that the transformation $H(\mathfrak{R}(V - F))$ have $H(\mathfrak{M}^-)$ for its closure (that the transformation $H(\mathfrak{R}(V^{-1} - F^*))$ have $H(\mathfrak{M}^+)$ for its closure).*

We leave the demonstration to the reader, pausing only to point out that $H(\mathfrak{M}^+) - iI$ and $H(\mathfrak{M}^-) + iI$ both have bounded inverses, each with domain $\mathfrak{H}$, so that $H(\mathfrak{M}^+)$ and $H(\mathfrak{M}^-)$ are necessarily closed.

As we shall show later, it is possible to have $\tilde{H}(V_1) \equiv \tilde{H}(V_2)$, $\tilde{V}_1 \neq \tilde{V}_2$, while $\tilde{H}(V_1)$ is maximal. In view of this situation, it is desirable to supplement Theorem 4.9 with the following proposition:

THEOREM 4.10. *Let S be an arbitrary maximal symmetric extension of H such that $\mathfrak{B}(S) \cdot \mathfrak{D}(A)$ has $\mathfrak{B}(S)$ for its closure and let S_A be the contraction of S with graph $\mathfrak{B}(S) \cdot \mathfrak{D}(A)$. Then $\tilde{S}_A \equiv S$ and there exists one and only one closed isometric transformation V with the following properties:*

(1) *the boundary condition $Af \, \varepsilon \, \mathfrak{R}(I - V)$ is nondegenerate;*

(2) *for every proper isometric extension V_1 from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ of V, the boundary condition $Af \, \varepsilon \, \mathfrak{R}(I - V_1)$ is degenerate;*

(3). $S_A \equiv H(V)$.

The relation $\tilde{S}_A \equiv S$ follows at once from the fact that the closure of the graph of S_A is the graph of S.

Now let $\mathfrak{N}$ be the closure of $A\mathfrak{B}(S_A)$. Then, by Theorem 2.2, there exists a closed isometric transformation V from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $\mathfrak{N} = \mathfrak{R}(I - V)$. Thus the condition $Af \, \varepsilon \, \mathfrak{R}(I - V)$ is nondegenerate and $S_A \equiv H(V)$, so that (1) and (3) are satisfied.

Now suppose $V \subset V_1$ and the condition $Af \, \varepsilon \, \mathfrak{R}(I - V_1)$ is nondegenerate. Then there must exist an element h^+ of $\mathfrak{D}(V_1)$ which is not in $\mathfrak{D}(V)$, such that $h^+ - Vh^+$ is in $\mathfrak{R}(A)$. Hence $H(V_1) \supset H(V)$, and, since $\tilde{H}(V)$ is maximal symmetric, $\tilde{H}(V_1) \equiv \tilde{H}(V)$. But then $S \equiv \tilde{H}(V_1)$ and, since $S_A \equiv H(V)$ and the graph of $H(V_1)$ is in $\mathfrak{D}(A)$, we have, by definition of S_A, $H(V) \supseteq H(V_1)$, which is incompatible with the inequality $H(V_1) \supset H(V)$. Consequently, we must conclude that the condition $Af \, \varepsilon \, \mathfrak{R}(I - V_1)$ is degenerate; therefore (2) is satisfied by the transformation V.

To conclude the proof, we have only to note that if V_1 is an arbitrary closed isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $S_A \equiv H(V_1)$, we have clearly $V_1 \supseteq V$; hence, since we have already shown that $V \subset V_1$ implies that the condition $Af \, \varepsilon \, \mathfrak{R}(I - V_1)$ is degenerate, there exists only one closed isometric transformation V satisfying conditions (1)–(3).

It is to be emphasized that the equations $\overline{\mathfrak{R}}(V - F) = \mathfrak{M}^-$, $\overline{\mathfrak{R}}(V^{-1} - F^*) = \mathfrak{M}^+$ are necessary but not sufficient, respectively, for the conditions

$$\tilde{H}(\mathfrak{R}(V - F)) = H(\mathfrak{M}^-), \qquad \tilde{H}(\mathfrak{R}(V^{-1} - F^*)) = H(\mathfrak{M}^+)$$

of Theorem 4.9. A portion of the interest of the following theorem derives from this fact.

THEOREM 4.11. *Let V be an isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$, and let F have the same meaning as in Theorem 3.11. Let K_V be the transformation $(I - VF^*)^{-1}(V - F)$. Then if $(\mathfrak{M}^- \ominus \overline{\mathfrak{R}}(K_V)) \cdot \mathfrak{M}_A^- = \mathfrak{O}$, $H(V)$ is maximal symmetric with deficiency index $(0, p)$ and if $(\mathfrak{M}^+ \ominus \overline{\mathfrak{D}}(K_V)) \cdot \mathfrak{M}_A = \mathfrak{O}$, $H(V)$ is maximal symmetric with deficiency index $(p, 0)$; furthermore, if X is the isometric transformation from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that*

$$\mathfrak{B}(H(V)) \cdot (\mathfrak{B}^+ + \mathfrak{B}^-) = \mathfrak{R}(I - X),$$

then $\mathfrak{D}(X) \cdot \mathfrak{B}_A^+$ is dense in $\mathfrak{D}(\tilde{X})$ and $\mathfrak{R}(X) \cdot \mathfrak{B}_A^-$ is dense in $\mathfrak{R}(\tilde{X})$.

Finally, all of the preceding statements remain true if K_V is the transformation $(V^{-1} - F^*)^{-1}(I - V^{-1}F)$.

Let h^+ be an arbitrary element of the domain of

$$K_V \equiv (I - VF^*)^{-1}(V - F).$$

and let $h^- = K_V h^+$. Then $(V-F)h^+ = k^-$ and $(I - VF^*)h^- = k^-$, where k^- is some element of $\mathfrak{M}^-$. Thus

$$Vh^+ - Fh^+ = h^- - VF^*h^-,$$

or

$$V(h^+ + F^*h^-) = h^- + Fh^+.$$

Therefore

$$h^+ + F^*h^- - h^- - Fh^+ = (h^+ - Fh^+) - (h^- - F^*h^-)$$

is in $\mathfrak{R}(I-V)$. Furthermore, by the definition of F and F^* in Theorem 3.11, $h^+ - Fh^+$ and $h^- - F^*h^-$ are in $\mathfrak{R}(A)$, $A_1^{-1}(h^+ - Fh^+)$ is in $\mathfrak{B}_A^-$, and $A_1^{-1}(h^- - F^*h^-)$ in $\mathfrak{B}_A^+$. Hence, if $\{f^-, -if^-\} = A_1^{-1}(h^+ - Fh^+)$ and $\{f^+, if^+\} = A_1^{-1}(h^- - F^*h^-)$, then $(h^+ - Fh^+) - (h^- - F^*h^-)$ is in $\mathfrak{R}(A) \cdot \mathfrak{R}(I-V)$ and

$$A_1^{-1}[(h^+ - Fh^+) - (h^- - F^*h^-)] = \{f^-, -if^-\} - \{f^+, if^+\}$$

is in $\mathfrak{B}(H(V))$. Thus $\{f^+, if^+\}$ is in $\mathfrak{D}(X)$ and $\{f^-, -if^-\} = X\{f^+, if^+\}$, where X has the meaning given in the theorem. In consequence of this result, we have the relations

(4.6) $$\mathfrak{D}(X) \cdot \mathfrak{B}_A^+ \supseteq A_1^{-1}(I - F^*)\mathfrak{R}(K_V),$$

(4.7) $$\mathfrak{R}(X) \cdot \mathfrak{B}_A^- \supseteq A_1^{-1}(I - F)\mathfrak{D}(K_V).$$

Now let $\{g^+, ig^+\}$ be an element of $\mathfrak{B}^+$ such that

$$(f^+, ig^+) - (if^+, g^+) = -i(\{f^+, if^+\}, \{g^+, ig^+\}) = 0$$

for all $\{f^+, if^+\}$ in $\mathfrak{D}(X) \cdot \mathfrak{B}_A^+$ and thus, in view of (4.6), for all $\{f^+, if^+\}$ which are in $A_1^{-1}(I - F^*)\mathfrak{R}(K_V)$. Then if $\{g^-, -ig^-\} = G\{g^+, ig^+\}$, we have

$$(f^+, ig^+ + ig^-) - (if^+, g^+ - g^-) = 0.$$

Hence, from our fundamental formula, since $\{g^+ - g^-, ig^+ + ig^-\}$ is in $\mathfrak{D}(A)$, we have

$$(A\{f^+, if^+\}, WA\{g^+ - g^-, ig^+ - ig^-\}) = 0,$$

for all $A\{f^+, if^+\}$ in $(I - F^*)\mathfrak{R}(K)$. But $WA\{g^+ - g^-, ig^+ - ig^-\}$ is in $\mathfrak{M}_A^-$, by Theorem 3.11, (3), and $\mathfrak{R}(F^*)$ is in $\mathfrak{M}^+ = \mathfrak{M} \ominus \mathfrak{M}^-$. Thus, if

$$(\mathfrak{M}^- \ominus \overline{\mathfrak{R}}(K_V)) \cdot \mathfrak{M}_A^- = \mathfrak{O},$$

we must have

$$WA\{g^+ - g^-, ig^+ - ig^-\} = 0,$$

whence it follows at once that $\{g^+, ig^+\} = 0$. Hence, if

$$(\mathfrak{M}^- \ominus \overline{\mathfrak{R}}(K_V)) \cdot \mathfrak{M}_A^- = \mathfrak{O},$$

then $\mathfrak{D}(X) \cdot \mathfrak{B}_A^-$ has $\mathfrak{B}^+$ for its closure. Therefore $\mathfrak{B}^+ = \mathfrak{D}(\tilde{X})$, and, by Theorems 2.2 and 4.2, $H(V)$ is maximal symmetric with deficiency index $(0, p)$, as we wished to prove. Moreover, $\mathfrak{R}(X) \cdot \mathfrak{B}_A^-$ is clearly $X[\mathfrak{D}(X) \cdot \mathfrak{B}_A^+]$ and therefore has $\mathfrak{R}(\tilde{X})$ for its closure; since we have already shown that $\mathfrak{D}(X)$ is the closure of $\mathfrak{D}(X) \cdot \mathfrak{B}_A^+$.

Thus our statements concerning the condition $(\mathfrak{M}^- \ominus \overline{\mathfrak{R}}(K_V) \cdot \mathfrak{M}_A^- = \mathfrak{O}$ are completely established, for $K_V \equiv (I - VF^*)^{-1}(V - F)$. Since the condition $(\mathfrak{M}^+ \ominus \overline{\mathfrak{D}}(K_V)) \cdot \mathfrak{M}_A^+ = \mathfrak{O}$ can be discussed along entirely similar lines, making use of the relation (4.7) instead of (4.6), we leave this portion of the argument to the reader.

Now let us suppose that $K_V \equiv (V^{-1} - F^*)^{-1}(I - V^{-1}F)$. Again let h^+ be an element of the domain of K_V and let $h^- = K_V h^+$. Then $(I - V^{-1}F)h^+ = k^+$ and $(V^{-1} - F^*)h^- = k^+$, where k^+ is some element of $\mathfrak{M}^+$. Then

$$h^+ - V^{-1}Fh^+ = V^{-1}h^- - F^*h^-,$$

or

$$V^{-1}(h^- + Fh^+) = h^+ + F^*h^-,$$

and

$$(h^- + Fh^+) = V(h^+ + F^*h^-).$$

Since this is an equation which occurs in the proof for the case

$$K_V \equiv (I - VF^*)^{-1}(V - F),$$

and which, together with arguments not involving K_V, leads to the relations (4.6) and (4.7), the rest of the demonstration now proceeds as before.

While we have not investigated all the questions involved, the indications are that Theorem 4.11 has no precise converse: as far as we can determine, it is possible to have $\tilde{H}(V) \equiv \tilde{H}(V_1)$, $V_1 \not\equiv V$, while V, but not V_1, satisfies one of the conditions of Theorem 4.10. We do, however, have the following theorem:

THEOREM 4.12. *Let S and S_A be as in Theorem 4.10, and let V be the isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ associated with S by that theorem. Let X be the isometric transformation from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that $\mathfrak{R}(I - X) = \mathfrak{B}(S_A) \cdot (\mathfrak{B}^+ + \mathfrak{B}^-)$, and let $\mathfrak{D}(X) \cdot \mathfrak{B}_A^+$ be dense in $\mathfrak{D}(X)$, or, equivalently, let $\mathfrak{R}(X) \cdot \mathfrak{B}_A^-$ be dense in $\mathfrak{R}(X)$. Let V_1 be an arbitrary maximal isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $V_1 \supseteq V$, and let K_{V_1} be either*

$$(I - V_1F^*)^{-1}(V_1 - F)$$

or

$$(V^{-1} - F^*)^{-1}(I - V^{-1}F)$$

according as $\mathfrak{D}(V_1) = \mathfrak{M}^+$ *or not. Then*

$$(\mathfrak{M}^- \ominus \overline{\mathfrak{R}(K_{V_1})}) \cdot \mathfrak{M}_A^- = \mathfrak{O}$$

if the deficiency index of S is $(0, p)$, and

$$(\mathfrak{M}^+ \ominus \overline{\mathfrak{D}(K_{V_1})}) \cdot \mathfrak{M}_A^+ = \mathfrak{O}$$

if the deficiency index of S is $(p, 0)$.

Let $\{f^+, if^+\}$ be an arbitrary element of $\mathfrak{D}(X) \cdot \mathfrak{B}_A^+$, and let

$$\{f^-, -if^-\} = X\{f^+, if^+\}.$$

Then $\{f^+ - f^-, if^+ + if^-\}$ is in $\mathfrak{D}(A)$ and

$$\begin{aligned} A\{f^+ - f^-, if^+ + if^-\} &= A\{f^+, if^+\} - A\{f^-, -if^-\} \\ &= -h^- + F^*h^- + h^+ - Fh^+, \end{aligned}$$

where h^- is in $\mathfrak{M}^-$, h^+ in $\mathfrak{M}^+$. But $\{f^+ - f^-, if^+ + if^-\}$ is in $\mathfrak{B}(S)$ also; so we have

$$h^- + Fh^+ = V_1(F^*h^- + h^+).$$

Thus, if $\mathfrak{D}(V_1) = \mathfrak{M}^+$,

$$h^- + Fh^+ = V_1 F^* h^- + V_1 h^+,$$

or, solving for h^-,

$$h^- = (I - V_1 F^*)^{-1}(V_1 - F)h^+ = K_{V_1} h^+.$$

On the other hand, if $\mathfrak{D}(V_1) \neq \mathfrak{M}^+$, we must have $\mathfrak{R}(V_1) = \mathfrak{M}^-$, so that

$$V_1^{-1} h^- + V_1^{-1} F h^+ = F^* h + h^+.$$

Thus, solving for h^-, we have in this case

$$h^- = (V_1^{-1} - F^*)^{-1}(I - V_1^{-1} F)h^+ = K_{V_1} h^+.$$

Therefore, in both cases, $h^- = K_{V_1} h^+$.

Now let k^- be an element of $\mathfrak{M}_A^-$ which is perpendicular to $\overline{\mathfrak{R}(K_{V_1})}$. Then, if $A_1^{-1} k^- = \{g^+ - g^-, ig^+ + ig^-\}$, we have

$$(f^+, ig^+ + ig^-) - (if^+, g^+ - g^-) = -(h^- - F^*h^-, Wk^-) = -i(h^- - F^*h^-, k^-),$$

where f^+ and h^- have the same meanings as above; and this equation is equiv-alent to

$$(f^+, ig^+) - (if^+, g^+) = -i(h^-, k^-),$$

since F^*h^- is in $\mathfrak{M}^+$. Hence, since k^- is perpendicular to $\overline{\mathfrak{R}(K_{V_1})}$,

$$(\{f^+, if^+\}, \{g^+, ig^+\}) = 0.$$

Now suppose that S has deficiency index $(0, p)$. Then $\mathfrak{D}(\tilde{X}) = \mathfrak{B}^+$. Hence, if $\mathfrak{D}(X) \cdot \mathfrak{B}_A^+$ is dense in $\mathfrak{D}(\tilde{X})$, we must have $\{g^+, ig^+\} = 0$, since $\{f^+, if^+\}$ is an arbitrary element in $\mathfrak{D}(X) \cdot \mathfrak{B}_A^+$. Therefore $k^- = 0$ and $(\mathfrak{M}^- \ominus \mathfrak{R}(K_{v_1})) \cdot \mathfrak{M}_A^- = \mathfrak{O}$, as we wished to prove.

On the other hand, if S has deficiency index $(p, 0)$, then $\mathfrak{R}(\tilde{X}) = \mathfrak{B}^-$; and an entirely similar argument leads to the conclusion $(\mathfrak{M}^+ \ominus \overline{\mathfrak{D}}(K_{v_1})) \cdot \mathfrak{M}_A^+ = \mathfrak{O}$.

Finally, we note that if $\mathfrak{R}(I - X) \subseteq \mathfrak{D}(A)$, then $X\{f^+, if^+\}$ is in $\mathfrak{B}_A^-$ whenever $\{f^+, if^+\}$ is in $\mathfrak{B}_A^+$. Hence $\mathfrak{R}(\tilde{X})$ is clearly the closure of $\mathfrak{R}(X) \cdot \mathfrak{B}_A^-$ whenever $\mathfrak{D}(\tilde{X})$ is the closure of $\mathfrak{D}(\tilde{X}) \cdot \mathfrak{B}_A^+$, as indicated in the theorem.

All of the Theorems 4.8–4.12 apply, of course, to the case in which A is bounded as well as to the case in which A is unbounded. For the former case, however, they are superficial, in view of Theorem 4.1, whereas in the latter they are not. The reason for this is to be found in Theorem 3.11, which tells us that the transformation F has a bound less than unity when A is bounded, and the bound unity which it never attains when A is unbounded. Consequently, if A is bounded and V is an arbitrary isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$, $V - F$ and $V^{-1} - F^*$ have bounded inverses. On the other hand, if A is unbounded, $(V - F)^{-1}$ and $(V^{-1} - F^*)^{-1}$ always exist, but are in general unbounded.

5. Pathology of unbounded reduction operators. We now examine certain pathological aspects of the behavior of discontinuous reduction operators, with a view to revealing more clearly the significance of the results of the preceding section.

To begin, we state and prove three lemmas, the roles of which will become clear later.

LEMMA 4.2. *Let $\mathfrak{D}$ be the domain, everywhere dense in a Hilbert space $\mathfrak{L}$, of a closed linear unbounded transformation R with range in a Hilbert space $\mathfrak{L}_0$. Then there exists a Hilbert space $\mathfrak{U}$ in $\mathfrak{L}$ such that $\mathfrak{U} \cdot \mathfrak{D} = \mathfrak{O}$.*

If R has range $\mathfrak{L}_0$ and R^{-1} exists, then the set of elements f of $\mathfrak{D}(R^*)$ for which R^*f is in $\mathfrak{L} \ominus \mathfrak{U}$, is dense in $\mathfrak{L}_0$.

According to a theorem previously noted, we can determine a self-adjoint transformation T in $\mathfrak{L}$, with domain $\mathfrak{D}$.† Furthermore, there must exist two Hilbert spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$, $\mathfrak{L}_1 + \mathfrak{L}_2 = \mathfrak{L}$, both of which reduce T. Let T_k be the transformation induced by T in $\mathfrak{L}_k$, $(k = 1, 2)$. Then at least one of the transformations T_1, T_2, say T_1, is unbounded. Let T_3 be the linear transformation in $\mathfrak{L}$ which is equal to T_1 in $\mathfrak{L}_1$, to I in $\mathfrak{L}_2$. Then T_3 is clearly unbounded self-adjoint and $\mathfrak{D}(T_3) \supseteq \mathfrak{D}(T)$, $\mathfrak{D}(T_3) \supseteq \mathfrak{L}_2$. Furthermore, there exists a unitary transformation U in $\mathfrak{L}$ such that $U\mathfrak{D}(T_3)$ and $\mathfrak{D}(T_3)$ intersect only in the

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† Murray, Theorem 1.24.

origin.† Hence the Hilbert space $\mathfrak{U} = U\mathfrak{L}_2$ intersects the manifold $\mathfrak{D}$ only in the origin.

To prove the final assertion of the lemma, we consider an arbitrary element g of $\mathfrak{L}_0$ which satisfies the equation $(g, f) = 0$ for all f in $\mathfrak{D}(R^*)$ such that R^*f is in $\mathfrak{L} \ominus \mathfrak{U}$; if R has range $\mathfrak{L}_0$ and R^{-1} exists, then $R^{-1}g$ is defined and $(R^{-1}g, R^*f) = 0$ for all R^*f in $\mathfrak{L} \ominus \mathfrak{U}$. But if R^{-1} exists and has domain $\mathfrak{L}_0$, it follows also that R^* has range $\mathfrak{L}$, and thus $\mathfrak{R}(R^*) \supset \mathfrak{L} \ominus \mathfrak{U}$. Hence $R^{-1}g$ is in $\mathfrak{D} \cdot \mathfrak{U}$ and thus $R^{-1}g = 0$. Consequently $g = 0$, and the proof is complete.

LEMMA 4.3. *Let Q be a unitary transformation in a Hilbert space $\mathfrak{L}$ such that $Q^2 + I \equiv O$; and let the characteristic manifolds $\mathfrak{L}^+$ and $\mathfrak{L}^-$ of Q, for the characteristic values $+i$ and $-i$, respectively, both be Hilbert spaces. Let D be an unbounded nonnegative definite self-adjoint transformation in $\mathfrak{L}$ such that D^{-1} exists and $D \equiv QD^{-1}Q^{-1}$. Then there exists a maximal Q-symmetric manifold $\mathfrak{N}$ in $\mathfrak{L}$, with the following properties:*

(1) $\mathfrak{N}$ is in $\mathfrak{D}(D)$ and $D\mathfrak{N}$ has a maximal Q-symmetric closure;

(2) for each cardinal number m on the range $1 \leq m \leq \aleph_0$, there exists in $\mathfrak{N}$ a closed linear Q-symmetric manifold $\mathfrak{N}_1$ which has the Q-deficiency index $(m, m+p)$ or $(m+p, m)$ according as $(0, p)$ or $(p, 0)$ is the Q-deficiency index of $\mathfrak{N}$, while $D\mathfrak{N}_1$ has the same closure as $D\mathfrak{N}$.

Let $\mathfrak{U}$ be the range of $E(1 - 0)$, where $E(\lambda)$ is the resolution of the identity in $\mathfrak{L}$ associated with D, and let D_1 be the transformation induced in $\mathfrak{U}$ by D. Then D_1^{-1} exists, $\mathfrak{U}$ is Q-symmetric, and in $Q\mathfrak{U}$, $D = QD_1^{-1}Q_1$, by Theorem 3.5. By the same theorem, D is equal to I in $\mathfrak{L} \ominus (\mathfrak{U} + Q\mathfrak{U})$, and hence D_1^{-1} is unbounded since D is. Therefore $\mathfrak{D}(D_1^{-1})$ is dense in $\mathfrak{U}$, but not identically $\mathfrak{U}$. Applying Lemma 4.2, we determine a Hilbert space $\mathfrak{U}_0$ in $\mathfrak{U}$ such that $\mathfrak{D}(D_1^{-1}) \cdot \mathfrak{U}_0 = \mathfrak{D}$. Then, by the second paragraph of the same lemma, $D(\mathfrak{U} \ominus \mathfrak{U}_0)$ is dense in $\mathfrak{U}$. Hence, if $\mathfrak{U}_1$ is any closed linear subspace of $\mathfrak{U}_0$, with dimension number m, $D(\mathfrak{U} \ominus \mathfrak{U}_1)$ is dense in $\mathfrak{U}$. Now let $\mathfrak{N}$ be an arbitrary maximal Q-symmetric extension of $\mathfrak{U}$, and let $\mathfrak{N}_1 = (\mathfrak{U} \ominus \mathfrak{U}_1) + (\mathfrak{N} \ominus \mathfrak{U})$. Then, since $\mathfrak{N} \ominus \mathfrak{U}$ is in the manifold $\mathfrak{L} \ominus (\mathfrak{N} + Q\mathfrak{N})$, where $D = I$, it follows that $D\mathfrak{N}$ and $D\mathfrak{N}_1$, both have the closure $\mathfrak{N}$. As $\mathfrak{N}_1$ evidently has the Q-deficiency index stated in the lemma, the proof is complete.

LEMMA 4.4. *Let $\mathfrak{L}$, $\mathfrak{L}^+$, $\mathfrak{L}^-$, Q, and D be as in Lemma 4.3. Let m be an arbitrary cardinal number on the range $0 \leq m \leq \aleph_0$. Then there exists a maximal Q-symmetric manifold $\mathfrak{N}$ in $\mathfrak{D}(D)$ such that $D\mathfrak{N}$ is a closed linear Q-symmetric manifold with the Q-deficiency index $(m, m+p)$ or $(m+p, m)$ according as $(0, p)$ or $(p, 0)$ is the Q-deficiency index of $\mathfrak{N}$.*

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† Von Neumann, Journal für die reine und angewandte Mathematik (Crelle), loc. cit.

Let $\mathfrak{U}$ and D_1 have the same meanings as in the proof of Lemma 4.3. Let V be the isometric transformation with domain in $\mathfrak{L}^+$ and range in $\mathfrak{L}^-$, such that $\mathfrak{U} = \mathfrak{R}(I - V)$. Then it is readily shown that D_1 determines a unique bounded self-adjoint transformation D_0 in $\mathfrak{D}(V)$, such that for each $h^+ - Vh^+$ in $\mathfrak{U}$,

$$D(h^+ - Vh^+) = D_0 h^+ - VD_0 h^+$$

Moreover, D_0 has the same bound, 1, as D_1, and D_0^{-1} exists and is unbounded because D_1^{-1} is unbounded.

Therefore, applying Lemma 4.2, we can determine a bounded closed contraction C of D_0 such that $\mathfrak{D}(V) \ominus \mathfrak{D}(C)$ has the dimension number m, $(0 \leqq m \leqq \aleph_0)$, and such that $\mathfrak{R}(C)$ is dense in $\mathfrak{D}(V)$. Hence C^{-1}, which exists because D_0^{-1} does, is a closed symmetric transformation in $\mathfrak{D}(V)$, and $R \equiv [(C^{-1})^* C^{-1}]^{1/2}$ is a self-adjoint transformation with the same range as $(C^{-1})^*$. Since $D_0^{-1} \supseteq C^{-1}$ and $D_0^{-1} \equiv (D_0^{-1})^*$, it follows that $D_0^{-1} \subseteq (C^{-1})^*$ and hence that $(C^{-1})^*$ has range identically $\mathfrak{D}(V)$. Thus R has range $\mathfrak{D}(V)$ and domain $\mathfrak{R}(C)$.† Consequently as R is self-adjoint, R^{-1} exists.

Now consider the set $\mathfrak{N}_1$ of all elements of $\mathfrak{L}$ which can be written in the form

$$(I + iR^{-1})h^+ - V(I - iR^{-1})h^+.$$

Since $I + iR^{-1} \equiv i(R^{-1} - iI)$, and R^{-1} is self-adjoint in $\mathfrak{D}(V)$, $(I + iR^{-1})$ has range identically $\mathfrak{D}(V)$ and $\mathfrak{N}_1$ is precisely the set of all elements of $\mathfrak{L}$ which can be written in the form

$$k^+ - V(I - iR^{-1})(I + iR^{-1})^{-1}k^+,$$

where k^+ is in $\mathfrak{D}(V)$. But $(I - iR^{-1})(I + iR^{-1})^{-1}$ is evidently a unitary transformation in $\mathfrak{D}(V)$ and thus $V(I - iR^{-1})(I + iR^{-1})^{-1}$ is an isometric transformation with domain $\mathfrak{D}(V)$ and range $\mathfrak{R}(V)$. Hence $\mathfrak{N}_1$ is closed Q-symmetric, by Theorem 2.2.

Next, let $\mathfrak{N}$ be any maximal Q-symmetric extension of $\mathfrak{N}_1$. Then $\mathfrak{N} \ominus \mathfrak{N}_1$ belongs to the manifold where $D = I$ and thus is in $\mathfrak{D}(D)$. Furthermore, every element

$$(I + iR^{-1})h^+ - V(I - iR^{-1})h^+$$

of $\mathfrak{N}_1$ can be expressed in the form

$$(h^+ - Vh^+) + i(R^{-1}h^+ + VR^{-1}h^+).$$

But $R^{-1}h^+$ is in the range of D_0 by definition of R^{-1} and $R^{-1}h^+ - VR^{-1}h^+$ is

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† Murray, Theorem 1.24.

thus in the range of D_1. Hence, since $\Re(I+V)=Q\Re$ and since, in $Q\Re$, $D=QD_1^{-1}Q^{-1}$, it follows that $R^{-1}h^+ + VR^{-1}h^+$ is in $\mathfrak{D}(D)$. Hence $\Re$ is in $\mathfrak{D}(D)$. Moreover, $D\Re$ and $D\Re_1$ are easily shown to be linear Q-symmetric.

Now, to determine the Q-deficiency index of $D\Re_1$, we again take account of the fact that $\Re_1$ is in $\mathfrak{U}+Q\mathfrak{U}=\mathfrak{D}(V)+\Re(V)$. Since this manifold reduces D, $D\Re_1$ also belongs to it; moreover, $\Re\ominus\Re_1$ reduces D since it belongs to $\mathfrak{L}\ominus(\mathfrak{D}(V)+\Re(V))$. Consequently, there exists a unique isometric transformation X with domain in $\mathfrak{D}(V)$ and range in $\Re(V)$ such that $D\Re_1=\Re(I-X)$; and the Q-deficiency index of $D\Re_1$ is precisely $(n, q+p)$ or $(n+p, q)$, where n and q are the dimension numbers of $\mathfrak{D}(V)\ominus\mathfrak{D}(X)$ and $\Re(V)\ominus\Re(X)$, respectively, according as $(0, p)$ or $(p, 0)$ is the Q-deficiency index of $\Re$. Furthermore, $D\Re$ is closed if and only if $D\Re_1$ is closed and $D\Re_1$ is closed if and only if X is closed.

Thus to complete the proof it remains only to be shown that X is closed and that $\mathfrak{D}(V)\ominus\mathfrak{D}(X)$ and $\Re(V)\ominus\Re(X)$ both have the dimension number m.

To determine X, we begin by analyzing further the manifold $D\Re_1$, using the resolution

$$h^+ - Vh^+ + i(R^{-1}h^+ + VR^{-1}h^+)$$

for an element of $\Re_1$. Since

$$D(h^+ - Vh^+) = D_0h^+ - VD_0h^+$$

and

$$iD(R^{-1}h^+ + VR^{-1}h^+) = i(D_0^{-1}R^{-1}h^+ + VD_0^{-1}R^{-1}h^+),$$

every element of $D\Re_1$ can be written in the form

$$(D_0 + iD_0^{-1}R^{-1})h^+ - V(D_0 - iD_0^{-1}R^{-1})h^+.$$

Conversely, every element of this form is easily shown to be in $D\Re_1$. Thus, since $(D_0+iD_0^{-1}R^{-1})h^+$ is in $\mathfrak{D}(V)$ and $V(D_0-iD_0^{-1}R^{-1})h^+$ is in $\Re(V)$, it follows that

$$\mathfrak{D}(X) = \Re(D_0 + D_0^{-1}R^{-1})$$

and that

$$X(D_0 + iD_0^{-1}R^{-1})h^+ = V(D_0 - iD_0^{-1}R^{-1})h^+.$$

Hence, as $(D_0+iD_0^{-1}R^{-1})^{-1}$ is readily shown to exist,

$$X \equiv V(D_0 - iD_0^{-1}R^{-1})(D_0 + iD_0^{-1}R^{-1})^{-1},$$

or, by straightforward algebraic calculation,

$$X \equiv -V(D_0^{-1}R^{-1}D_0^{-1} + iI)(D_0^{-1}R^{-1}D_0^{-1} - iI)^{-1},$$

and

$$- V^{-1}X \equiv (D_0^{-1}R^{-1}D_0^{-1} + iI)(D_0^{-1}R^{-1}D_0^{-1} - iI)^{-1}.$$

We now recall that D_0^{-1} has range $\mathfrak{D}(V)$ and that R^{-1} has domain $\mathfrak{D}(V)$, while $\mathfrak{D}(R) = \mathfrak{R}(R^{-1})$ is in $\mathfrak{D}(D_0^{-1})$. Thus, it is evident that $D_0^{-1}R^{-1}D_0^{-1}$ is symmetric because its domain is dense in $\mathfrak{D}(V)$, and D_0^{-1} and R^{-1} are both self-adjoint. Furthermore, $- V^{-1}X$ is the Cayley transform of $D_0^{-1}R^{-1}D_0^{-1}$ from the equation at the end of the preceding paragraph. Therefore, X is closed if and only if $D_0^{-1}R^{-1}D_0^{-1}$ is closed and, if X is closed, the dimension numbers of $\mathfrak{D}(V)\ominus\mathfrak{D}(X)$ and $\mathfrak{R}(V)\ominus\mathfrak{R}(X)$ are n and q, respectively, where (n, q) is the deficiency index of $D_0^{-1}R^{-1}D_0^{-1}$. Consequently, we can complete the proof by showing that $D_0^{-1}R^{-1}D_0^{-1}$ is closed and that its deficiency index is (m, m).

To prove the first we begin by observing that

$$\mathfrak{R}(D_0^{-1}R^{-1}D_0^{-1}) = D_0^{-1}\mathfrak{D}(R)$$

is a closed linear manifold by definition of R. We next observe that D_0^{-1} and R^{-1} are positive definite, each with lower bound 1; in consequence, we have

$$(D_0^{-1}R^{-1}D_0^{-1}h^+, h^+) = (R^{-1}D_0^{-1}h^+, D_0^{-1}h^+) \geqq (D_0^{-1}h^+, D_0^{-1}h^+) \geqq \left| h^+ \right|^2.$$

Thus $D_0^{-1}R^{-1}D_0^{-1}$ is positive definite with lower bound greater than or equal to 1. Hence it is easily shown that $(D_0^{-1}R^{-1}D_0^{-1})^{-1}$ exists and is bounded. But, as we have already pointed out, $D_0^{-1}R^{-1}D_0^{-1}$ has a closed range; hence its inverse is closed and in consequence $D_0^{-1}R^{-1}D_0^{-1}$ itself is closed.

We have now only to show that the deficiency index of $D_0^{-1}R^{-1}D_0^{-1}$ is (m, m). Let us suppose that the deficiency index of $D_0^{-1}R^{-1}D_0^{-1}$ is (n, n); that it is of this form follows from the fact that $D_0^{-1}R^{-1}D_0^{-1}$ is positive definite. Also from this fact, it follows that $D_0^{-1}R^{-1}D_0^{-1}$ has a self-adjoint extension T with bounded inverse; indeed, T may be chosen with the same lower bound, 1, as $D_0^{-1}R^{-1}D_0^{-1}$. Furthermore, from Theorems 4.2, 2.8, and 2.2, it is readily deduced that n is the dimension number of

$$\mathfrak{B}(T) \ominus \mathfrak{B}(D_0^{-1}R^{-1}D_0^{-1}).$$

But from the fact that T^{-1} and $(D_0^{-1}R^{-1}D_0^{-1})^{-1}$ both exist and are bounded, it follows by a straightforward argument that

$$\mathfrak{B}(T) \ominus \mathfrak{B}(D_0^{-1}R^{-1}D_0^{-1})$$

and

$$\mathfrak{R}(T) \ominus \mathfrak{R}(D_0^{-1}R^{-1}D_0^{-1})$$

have the same dimension number. Moreover, $\mathfrak{R}(D_0^{-1}R^{-1}D_0^{-1}) = \mathfrak{D}(C)$ by definition of $\mathfrak{R}$, and $\mathfrak{R}(T) = \mathfrak{D}(V)$, since T is self-adjoint in $\mathfrak{D}(V)$, with T^{-1}

bounded. Thus, since $\mathfrak{D}(V)\ominus\mathfrak{D}(C)$ has the dimension number m by choice of C, $n=m$ and the proof is complete.

THEOREM 4.13. *Let A be an unbounded reduction operator for H^*. Then there exists a maximal isometric transformation V from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ with the following properties: (1) the boundary condition $Af \, \varepsilon \, \mathfrak{R}(I-V)$ is nondegenerate and $H(V)$ is maximal symmetric; (2) for an arbitrary cardinal number on the range $1 \leq m \leq \aleph_0$, there exists an isometric transformation V_1 from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $\tilde{V}_1 \equiv V$ and such that $H(V_1)$ has the deficiency index $(m, m+p)$ or $(m+p, m)$ according as $(0, p)$ or $(p, 0)$ is the deficiency index of $H(V)$.*

Let $B \equiv (A_1^*A_1)^{1/2}$. Then $A_1 \equiv XB$ where X is isometric with domain $\mathfrak{B}^+ + \mathfrak{B}^-$ and range $\mathfrak{M}.$† Applying Lemma 4.3, and the fact, stated in Theorem 3.6, that $B \equiv QB^{-1}Q^{-1}$, we conclude that there exists a maximal Q-symmetric manifold $\mathfrak{N}$ in $\mathfrak{B}^+ + \mathfrak{B}^-$ with the following properties: $\mathfrak{N} \subset \mathfrak{D}(B) = \mathfrak{D}(A)$ and $B\mathfrak{N}$ has a maximal Q-symmetric closure; $\mathfrak{N} \supset \mathfrak{N}_1$, where $\mathfrak{N}_1$ is closed linear Q-symmetric with Q-deficiency index $(m, m+p)$ or $(m+p, m)$ according as $(0, p)$ or $(p, 0)$ is the Q-deficiency index of $\mathfrak{N}$, and $B\mathfrak{N}_1$ has the same closure as $B\mathfrak{N}$.

Furthermore, since, by Theorem 3.3, $X\mathfrak{B}^+ = \mathfrak{M}^-$ and $X\mathfrak{B}^- = \mathfrak{M}^+$, it is clear that $XB\mathfrak{N} = A\mathfrak{N}$ has a maximal W-symmetric closure and that $XB\mathfrak{N}_1 = A\mathfrak{N}_1$ has the same closure as $A\mathfrak{N}$. Now let V_2 and V_1 be the isometric transformations from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ which correspond to $A\mathfrak{N}$ and $A\mathfrak{N}_1$, respectively, in accordance with Theorem 2.2, and let $V \equiv \tilde{V}_2$. Then V is maximal isometric from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ and $\tilde{V}_1 \equiv V$, since $(\overline{A\mathfrak{N}_1}) = \mathfrak{R}(I-V)$. Moreover, $H(V)$ and $H(V_1)$ have for their graphs $\mathfrak{B} + \mathfrak{N}$ and $\mathfrak{B} + \mathfrak{N}_1$, respectively. Hence, by Theorem 4.2, $H(V)$ is maximal symmetric and $H(V_1)$ has the deficiency index stated in the theorem. Thus, since the boundary conditions $Af \, \varepsilon \, \mathfrak{R}(I-V)$, $Af \, \varepsilon \, \mathfrak{R}(I-V_1)$ are obviously nondegenerate, the proof is complete.

THEOREM 4.14. *Let A be unbounded. Then there exists a maximal isometric transformation V from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ with the following properties: (1) the boundary condition $Af \, \varepsilon \, \mathfrak{R}(I-V)$ is nondegenerate and $H(V)$ has a maximal symmetric closure; (2) if m is an arbitrary cardinal number on the range $1 \leq m \leq \aleph_0$ and V has the W-deficiency index $(0, p)$ $((p, 0))$, there exists an isometric transformation V_1, $V_1 \subset V$, with W-deficiency index $(m, m+p)$ $((m+p, m))$ such that the boundary condition $Af \, \varepsilon \, \mathfrak{R}(I-V_1)$ is nondegenerate and such that $\tilde{H}(V_1) \equiv \tilde{H}(V)$.*

Let B and X have the same meanings as in the proof of Theorem 4.13. Then $B^{-1} \equiv QBQ^{-1}$. Let $\mathfrak{N}$ be a manifold in $\mathfrak{D}(B^{-1})$ which satisfies the conditions of Lemma 4.3 with $D \equiv B^{-1}$. Then $X\mathfrak{N}$ is clearly maximal W-symmetric

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† Murray, Theorem 1.24.

in $\mathfrak{M}$, and if $\mathfrak{N}_1$ has the same meaning as in Lemma 4.3, (2), then $X\mathfrak{N}_1$ has the W-deficiency index $(m, m+p)$ or $(m+p, m)$ according as $X\mathfrak{N}$ has the W-deficiency index $(0, p)$ or $(p, 0)$. Furthermore, $\mathfrak{N}$ and $\mathfrak{N}_1$ both belong to the range of B; hence $X\mathfrak{N}$ and $X\mathfrak{N}_1$ belong to the range of A. Therefore the boundary conditions $Af \mathrel{\varepsilon} X\mathfrak{N}_1$, $Af \mathrel{\varepsilon} X\mathfrak{N}$ are nondegenerate.

Now let V and V_1 be the isometric transformations from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ corresponding to $X\mathfrak{N}$ and $X\mathfrak{N}_1$ in accordance with Theorem 2.2. Then $H(V)$ and $H(V_1)$ have for their graphs $\mathfrak{B}+B^{-1}\mathfrak{N}$ and $\mathfrak{B}+B^{-1}\mathfrak{N}_1$, respectively. Furthermore, by Lemma 4.3, $B^{-1}\mathfrak{N}$ and $B^{-1}\mathfrak{N}_1$ have the same closure and the latter is maximal Q-symmetric in $\mathfrak{B}^++\mathfrak{B}^-$. Hence, by Theorem 4.2, $H(V)$ is maximal symmetric. Thus, since $X\mathfrak{N}$ is maximal W-symmetric, V is maximal isometric from $\mathfrak{M}^+$ to $\mathfrak{M}^-$, and V and V_1 have the properties described in Theorem 4.14.

In view of Theorem 4.14, the role of the condition (2) of Theorem 4.10 is now more clearly indicated.

THEOREM 4.15. *Let A be an unbounded reduction operator for H^*. Let m be an arbitrary cardinal number on the range $0 \leq m \leq \aleph_0$. Then there exists a maximal isometric transformation V from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ with the following properties:*

(1) *the boundary condition $Af \mathrel{\varepsilon} \mathfrak{R}(I-V)$ is nondegenerate;*

(2) *$H(V)$ is closed with the deficiency index $(m+p, m)$ or $(m, m+p)$ according as $(0, p)$ or $(p, 0)$ is the W-deficiency index of V;*

(3) *for every maximal symmetric extension S of $H(V)$, $S_A \equiv H(V)$, where S_A has the same meaning as in Theorem 4.5.*

Again, let B have the same meaning as in the proof of Theorem 4.13; as we have already noted, $B^{-1} \equiv QBQ^{-1}$. Hence, by Lemma 4.4, there exists a maximal Q-symmetric manifold $\mathfrak{N}$ in $\mathfrak{D}(B^{-1})$ such that $B^{-1}\mathfrak{N}$ is closed linear Q-symmetric with Q-deficiency index $(m, m+p)$ or $(m+p, m)$ according as $\mathfrak{N}$ has Q-deficiency index $(0, p)$ or $(p, 0)$. Furthermore, if $A \equiv XB$, the relations $\mathfrak{M}^+ = X\mathfrak{B}^-$, $\mathfrak{M}^- = X\mathfrak{B}^+$ imply that $X\mathfrak{N} = A(B^{-1}\mathfrak{N})$ has the W-deficiency index $(p, 0)$ or $(0, p)$ according as $\mathfrak{N}$ has the Q-deficiency index $(0, p)$ or $(p, 0)$.

Now let V be isometric from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $X\mathfrak{N} = \mathfrak{R}(I-V)$. Then, by definition, the W-deficiency index of V is the same as that of $X\mathfrak{N}$. Moreover, the graph of $H(V)$ is precisely $\mathfrak{B}+A_1^{-1}X\mathfrak{N} = \mathfrak{B}+B^{-1}\mathfrak{N}$. Thus, by Theorem 4.2, $H(V)$ is closed with deficiency index $(m, m+p)$ or $(m+p, m)$ according as $(p, 0)$ or $(0, p)$ is the W-deficiency index of V, and the boundary condition $Af \mathrel{\varepsilon} \mathfrak{R}(I-V)$ is obviously nondegenerate.

We are therefore left to prove only that $H(V)$ has the property (3). To do this, we consider an arbitrary maximal symmetric extension S of $H(V)$ and

the transformation S_A described in Theorem 4.5. Then $A\mathfrak{B}(S_A)$ is obviously a linear W-symmetric manifold in $\mathfrak{M}$; moreover, $A\mathfrak{B}(S_A) \supseteq A\mathfrak{B}(H(V))$ $= \mathfrak{R}(I-V) = \mathfrak{N}$. But $\mathfrak{N}$ is maximal W-symmetric; therefore we must have $A\mathfrak{B}(S_A) = \mathfrak{N}$, which is possible if and only if $S_A \equiv H(V)$. Thus the proof is complete.

We thus see that the condition that V be maximal isometric from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ is not sufficient either for $H(V)$ or for $\tilde{H}(V)$ to be maximal symmetric, even when the additional restriction that the condition $Af \varepsilon \mathfrak{R}(I-V)$ be nondegenerate is imposed. It is therefore natural to ask what properties $H(V)$ has under these circumstances. Without giving the proofs, we shall merely set down the following observations which the reader may verify: If V is maximal isometric from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ with W-deficiency index $(0, p)$ $((p, 0))$, if $\mathfrak{R}(I-V)$ belongs to $\mathfrak{R}(A)$, and if X denotes the isometric transformation from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that

$$\mathfrak{R}(I - X) = (\mathfrak{B}^+ + \mathfrak{B}^-) \cdot \mathfrak{B}(H(V));$$

then $\mathfrak{R}(X-G) \supseteq \mathfrak{B}_A^-$ $(\mathfrak{R}(X^{-1}-G^*) \supseteq \mathfrak{B}_A^+)$, G having here the same meaning as in Theorem 3.11; conversely, if X is a closed isometric transformation from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that $\mathfrak{D}(A) \cdot \mathfrak{R}(I-X)$ has $\mathfrak{R}(I-X)$ for its closure and if $\mathfrak{R}(X-G) \supseteq \mathfrak{B}_A^-$ $(\mathfrak{R}(X^{-1}-G^*) \supseteq \mathfrak{B}_A^+)$, then $A[\mathfrak{D}(A) \cdot \mathfrak{R}(I-X)]$ is a maximal W-symmetric manifold $\mathfrak{N}$ in $\mathfrak{M}$ with W-deficiency index $(0, p)$ $((p, 0))$. The reader will note that this proposition constitutes a slightly modified analogue of Theorem 4.8. In precisely the same sense, one can formulate valid analogues of Theorems 4.9–4.12.

We shall now show that the condition that V be maximal isometric from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ and that $Af \varepsilon \mathfrak{R}(I-V)$ be nondegenerate, is unnecessary as well as insufficient for either $H(V)$ or $\tilde{H}(V)$ to be maximal symmetric.

THEOREM 4.16. *Let A be an unbounded reduction operator for H^*. Let m be an arbitrary cardinal number on the range $0 \leq m \leq \aleph_0$. Then there exists a maximal symmetric extension S of H with the following properties:*

  (1) $\mathfrak{B}(S) \subset \mathfrak{D}(A)$;

  (2) $A\mathfrak{B}(S)$ *is a closed W-symmetric manifold in $\mathfrak{M}$ with W-deficiency index* $(m+p, m)$ *or* $(m, m+p)$ *according as S has the deficiency index $(0, p)$ or $(p, 0)$;*

  (3) *if $\mathfrak{N}$ is an arbitrary W-symmetric manifold in $\mathfrak{M}$ such that $\mathfrak{N} \supset A\mathfrak{B}(S)$, the boundary condition $Af \varepsilon \mathfrak{N}$ is degenerate.*

Again we introduce the transformation B used in the proof of Theorem 4.13. Then, in accordance with Lemma 4.4, there exists a maximal Q-symmetric manifold $\mathfrak{U}$ in $\mathfrak{D}(B) = \mathfrak{D}(A_1)$ such that $B\mathfrak{U}$ is closed and has the Q-deficiency index $(m+p, m)$ or $(m, m+p)$ according as $(0, p)$ or $(p, 0)$ is the Q-deficiency index of $\mathfrak{U}$. Thus $\mathfrak{B}+\mathfrak{U}$ is the graph of a maximal symmetric

extension S of H and the deficiency index of S is the Q-deficiency index of $\mathfrak{U}$, by Theorem 4.2; moreover, $\mathfrak{B}(S) = \mathfrak{B} + \mathfrak{U} \subset \mathfrak{D}(A)$.

Now let X be the isometric transformation from $\mathfrak{B}^+ + \mathfrak{B}^-$ to $\mathfrak{M}$ such that $A \equiv XB$. Then, as we know, $X\mathfrak{B}^+ = \mathfrak{M}^-$, $X\mathfrak{B}^- = \mathfrak{M}^+$, and $W \equiv XQX^{-1}$. Therefore $XB\mathfrak{U} = A\mathfrak{U} = A\mathfrak{B}(S)$ is closed and has the W-deficiency index $(m, m+p)$ or $(m+p, m)$ according as $(m+p, m)$ or $(m, m+p)$ is the Q-deficiency index of $B\mathfrak{N}$, and consequently according as $(p, 0)$ or $(0, p)$ is the deficiency index of S. Thus S has the properties (1) and (2) of the theorem.

Now let $\mathfrak{N}$ be a W-symmetric manifold in $\mathfrak{M}$ such that $A\mathfrak{B}(S) \subset \mathfrak{N}$. Then, clearly $H(\mathfrak{N}) \supseteq S$ and, since S is maximal symmetric and $H(\mathfrak{N})$ symmetric, this is possible if and only if $H(\mathfrak{N}) \equiv S$. Thus $\mathfrak{R}(A) \cdot \mathfrak{N} = A\mathfrak{B}(S)$ while $A\mathfrak{B}(S)$ is closed, and therefore the boundary condition $Af \, \varepsilon \, \mathfrak{N}$ is obviously degenerate, so that S satisfies (3).

Finally, to conclude this section, we prove a theorem which establishes unequivocally that the dependence on the operator A of the conditions given in Theorems 4.8, 4.9, and 4.11 is not in any sense superficial.

THEOREM 4.17. *Let $\mathfrak{M}$ be an arbitrary Hilbert space, and let $\mathfrak{M}^+$ and $\mathfrak{M}^-$ be Hilbert spaces in $\mathfrak{M}$ such that $\mathfrak{M} \ominus \mathfrak{M}^+ = \mathfrak{M}^-$. Let W be the linear transformation in $\mathfrak{M}$ such that $W = iI$ on $\mathfrak{M}^+$, $W = -iI$ on $\mathfrak{M}^-$. Let H be a closed linear symmetric transformation in a Hilbert space $\mathfrak{H}$, with deficiency index $(\aleph_0, \aleph_0)$.*

Then, if V is an arbitrary closed isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$, there exists a reduction operator C for H^*, with range-space $\mathfrak{M}$ and associated unitary transformation W in $\mathfrak{M}$, such that the boundary condition $Cf \, \varepsilon \, \mathfrak{R}(I - V)$ is nondegenerate and defines a linear symmetric extension of H which is not maximal and does not have a maximal closure. The operator C can be constructed so as to be either of type I or of type II.

If $\mathfrak{D}(V)$ has the dimension number $\aleph_0$, there exists a second reduction operator D for H^*, with range-space $\mathfrak{M}$ and associated unitary transformation W in $\mathfrak{M}$, such that the boundary condition $Df \, \varepsilon \, \mathfrak{R}(I - V)$ is nondegenerate and defines a maximal symmetric extension of $\bar{H}$. Here D can be chosen either of type I or of type II.

Our proof of this theorem requires that we first review the proof of Lemma 4.4, in order to take account of certain facts which are implicit there and which, for present purposes, must be formally stated. We observe first that the manifold $\mathfrak{N}$ specified in the lemma is in the proof determined as an arbitrary maximal Q-symmetric extension of a Q-symmetric manifold $\mathfrak{N}_1$ in $\mathfrak{L} \ominus \mathfrak{P}$, where $\mathfrak{P}$ is the manifold in which $Q = I$; and that $\mathfrak{N}_1$ is hypermaximal Q_1-symmetric in $\mathfrak{L} \ominus \mathfrak{P}$, where Q_1 is the contraction of Q with domain $\mathfrak{L} \ominus \mathfrak{P}$. Accordingly, if the transformation D of Lemma 4.4 is such that $\mathfrak{P} \cdot \mathfrak{L}^+ = \mathfrak{O}$

while $\mathfrak{P}\cdot\mathfrak{L}^-$ has dimension number p, then $\mathfrak{N}$ has Q-deficiency index $(0, p)$. Similarly, if $\mathfrak{P}\cdot\mathfrak{L}^-=\mathfrak{O}$, while $\mathfrak{P}\cdot\mathfrak{L}^+$ has the dimension number p, then $\mathfrak{N}$ has the Q-deficiency index $(p, 0)$. Furthermore, if $\mathfrak{P}\cdot\mathfrak{L}^+$ and $\mathfrak{P}\cdot\mathfrak{L}^-$ both have the dimension number $\aleph_0$, then $\mathfrak{N}$ can be chosen with arbitrary Q-deficiency index $(0, p)$ or $(p, 0)$. These facts we hold in reserve for application later.

Now let V_1 be an arbitrary maximal isometric extension from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ of the transformation V of the theorem to be proved, and let V_1 have W-deficiency index $(0, p)$ $((p, 0))$; further, let A_1 be a nonnegative definite self-adjoint transformation in $\mathfrak{B}^++\mathfrak{B}^-$, such that $A_1\equiv QA_1^{-1}Q^{-1}$, where $\mathfrak{B}^+$, $\mathfrak{B}^-$, and Q have the usual meanings, with reference to the transformation H of Theorem 4.17. Then, by Theorem 3.1 the linear transformation A with domain $\mathfrak{B}(H)+\mathfrak{D}(A_1)$, which is equal to 0 on $\mathfrak{B}(H)$ and to A_1 on $\mathfrak{D}(A_1)$ is a reduction operator for H^*. We consider two distinct cases: (1) A is unbounded of type I with characteristic index $(p, 0)$ $((0, p))$; (2) A is unbounded of type II and the intersections of $\mathfrak{B}^+$ and $\mathfrak{B}^-$ with the manifold $\mathfrak{P}$ on which $A_1=I$ are both Hilbert spaces. Since H has deficiency index $(\aleph_0, \aleph_0)$, the results of Chapter III, §2 assure the existence of reduction operators A satisfying either requirement.

We now apply Lemma 4.4 to establish the existence in $\mathfrak{D}(A_1^{-1}) = \mathfrak{R}(A_1)$ of a maximal Q-symmetric manifold $\mathfrak{N}$ such that $A_1^{-1}\mathfrak{N}$ is closed linear Q-symmetric but not maximal; here we of course employ the relation $A_1^{-1}\equiv QA_1Q^{-1}$. Moreover, taking account of the paragraph immediately following Theorem 4.17, we see that under case (1) of the preceding paragraph, $\mathfrak{N}$ has the Q-deficiency index $(p, 0)$ $((0, p))$, while under case (2), $\mathfrak{N}$ can be chosen with the same Q-deficiency index. In the latter case, we assume that $\mathfrak{N}$ is so chosen.

Applying Theorem 2.2, we next introduce the isometric transformation U from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that $\mathfrak{N}=\mathfrak{R}(I-U)$. Then U has, under either of the cases (1) and (2), the Q-deficiency index $(p, 0)$ $((0, p))$. Thus, by definition, $\mathfrak{B}^-\ominus\mathfrak{R}(U)$ and $\mathfrak{B}^+\ominus\mathfrak{D}(U)$ have the same dimension numbers as $\mathfrak{M}^+\ominus\mathfrak{D}(V_1)$ and $\mathfrak{M}^-\ominus\mathfrak{R}(V_1)$, respectively. Consequently, $\mathfrak{D}(U)$, $\mathfrak{R}(U)$, $\mathfrak{D}(V_1)$, and $\mathfrak{R}(V_1)$ being Hilbert spaces, we can define an isometric transformation Y with domain $\mathfrak{B}^++\mathfrak{B}^-$ and range $\mathfrak{M}$ such that $Y\mathfrak{B}^- = \mathfrak{M}^+$, $Y\mathfrak{B}^+ = \mathfrak{M}^-$, $Y\mathfrak{R}(U) = \mathfrak{D}(V_1)$, and $Y\mathfrak{D}(U) = \mathfrak{R}(V_1)$. Thus $Y\mathfrak{N} = \mathfrak{R}(I-V_1)$.

But, by Theorem 3.3, $C\equiv YA$ is a reduction operator for H^*, with range-space $\mathfrak{M}$, and associated transformation W in $\mathfrak{M}$, since $YQ^{-1}Y^{-1}$ is evidently identical with W. Furthermore, the condition $Cf \varepsilon \mathfrak{R}(I-V_1)$ clearly defines the same symmetric extension of H as the condition $Af \varepsilon \mathfrak{R}(I-U)$, that is, the extension S whose graph is $\mathfrak{B}+A_1^{-1}\mathfrak{N}$; and S is clearly closed linear but not maximal, since $A_1^{-1}\mathfrak{N}$ is a closed linear manifold and not maximal Q-symmetric. The argument here is, of course, essentially that used to prove Theorem

4.15. Hence, the condition $Cf \, \varepsilon \, \Re(I - V)$ defines an extension T of H which does not have a maximal symmetric closure, since we obviously have $\tilde{T} \subseteq S$. Moreover, since $\Re(I - U) \subset \Re(A)$, we have

$$\Re(I - V) \subseteq \Re(I - V_1) \subset \Re(C),$$

so that the condition $Cf \, \varepsilon \, \Re(I - V)$ is nondegenerate. Finally, since we have shown that A can be either of type I or type II, C can be of either type also, by Theorem 3.16. Thus the first portion of the theorem is completely proved.

We turn now to the second, and denote by (n, q) the W-deficiency index of V. For purposes of simplification, we assume $n \geqq q$; the alternative possibility can be discussed along lines entirely similar to those which we pursue in this case.

Setting $p = n - q$, we apply Theorems 3.5 and 3.6 to construct a reduction operator A for H^* such that A_1 is a nonnegative definite self-adjoint transformation in $\mathfrak{B}^+ + \mathfrak{B}^-$ and such that the manifold on which $A_1 = I$ is a linear manifold in $\mathfrak{B}^-$ with dimension number p. We then apply Lemma 4.4 and the facts stated in the paragraph immediately following Theorem 4.17 to determine a maximal Q-symmetric manifold $\mathfrak{N}$ in $\mathfrak{B}^+ + \mathfrak{B}^-$ such that $A_1\mathfrak{N}$ is closed linear Q-symmetric with the Q-deficiency index $(q, p+q)$.

Next, we introduce the isometric transformation U from $\mathfrak{B}^+$ to $\mathfrak{B}^-$ such that $A_1\mathfrak{N} = \Re(I - U)$. Then, by definition, $\mathfrak{B}^+ \ominus \mathfrak{D}(U)$ and $\mathfrak{B}^- \ominus \Re(U)$ have the dimension numbers q and $p+q = n$, respectively. Consequently, there exists an isometric transformation Y from $\mathfrak{B}^+ + \mathfrak{B}^-$ to $\mathfrak{N}$ such that

$$Y\mathfrak{B}^+ = \mathfrak{N}^-, \quad Y\mathfrak{B}^- = \mathfrak{N}^+, \quad Y\mathfrak{D}(U) = \Re(V), \quad Y\Re(U) = \mathfrak{D}(V).$$

Thus $Y\Re(I - U) = \Re(I - V)$.

But, again by Theorem 3.3, $D \equiv YA$ is a reduction operator for H^*, with range-space $\mathfrak{N}$ and associated transformation W in $\mathfrak{N}$; furthermore, $\Re(I - V)$ clearly belongs to $\Re(D)$, so that the boundary condition $Df \, \varepsilon \, \Re(I - V)$ is non-degenerate. Therefore, since this boundary condition obviously defines a symmetric extension S of H with graph $\mathfrak{B} + \mathfrak{N}$ and $\mathfrak{N}$ is maximal Q-symmetric, the first assertion of the second paragraph of the theorem now follows at once from Theorem 4.2. Finally, since it is evident from Theorem 3.13 that A can be chosen either of type I or of type II, subject to our previous conditions, we conclude from Theorem 3.16 that D can be of either type.

6. A special class of boundary conditions. From Theorem 4.17 we of course conclude that the general result of Theorem 4.8 admits of no effective simplification. Consequently, in dealing with any particular concrete unbounded reduction operator A, one would naturally look for special properties of A which might lead to less general but more readily applicable results. In

this connection, both for its illustrative value and its usefulness in connection with certain applications of the theory here developed, we now consider a specialization of the general situation which we have heretofore been studying and subject it to study by methods of a quite different sort from those employed in §4.

We require a preliminary lemma.

LEMMA 4.5. *Let R be a transformation in a Hilbert space $\mathfrak{L}$, with at least one point μ in its resolvent set, and let $(R-\mu I)^{-1}$ be totally continuous. Then, if T is an arbitrary bounded transformation in $\mathfrak{L}$ and λ belongs to the resolvent set of $R+T$, $(R+T-\lambda I)^{-1}$ is totally continuous. In particular, $(R-\lambda I)^{-1}$ is totally continuous for every point λ in the resolvent set of R.*

We consider an arbitrary set $\mathfrak{U}$ in $\mathfrak{R}(R+T-\lambda I)=\mathfrak{L}$, λ being in the resolvent set of $R+T$. Then, if μ is in the resolvent set of R,

$$\mathfrak{D}(R - \mu I) = \mathfrak{D}(R + T - \lambda I) = \mathfrak{R}((R + T - \lambda I)^{-1})$$

and

$$(R + T - \lambda I)^{-1}\mathfrak{U} = (R - \mu I)^{-1}(R - \mu I)(R + T - \lambda I)^{-1}\mathfrak{U}$$

$$= (R - \mu I)^{-1}[(- T + (\lambda - \mu)I)(R + T - \lambda I)^{-1} + I]\mathfrak{U}.$$

Now suppose $\mathfrak{U}$ is a bounded set. Then, since

$$[(- T + (\lambda - \mu)I)(R + T - \lambda I)^{-1}]$$

is a bounded transformation,

$$[(- T + (\lambda - \mu)I)(R + T - \lambda I)^{-1} + I]\mathfrak{U}$$

is a bounded set. Thus, if $(R-\mu I)^{-1}$ is totally continuous, it follows from the above equation that $(R+T-\lambda I)^{-1}\mathfrak{U}$ is compact. Thus $(R+T-\lambda I)^{-1}$ is totally continuous, as we wished to show. Since, in particular, we can take $T \equiv 0$, the last statement of the lemma follows immediately, and the proof is complete.

THEOREM 4.18. *Let H have the deficiency index (n, n), and let A be a reduction operator of the sort described in Theorem 3.17, with range-space $\mathfrak{L} \oplus \mathfrak{L}$. Let S be the self-adjoint transformation in $\mathfrak{H} \oplus \mathfrak{L}$ which corresponds to A in accordance with Theorem 3.17. Let $\mathfrak{D}^*$ be the set of elements f of $\mathfrak{D}^*$ such that $\{f, H^*f\}$ is in $\mathfrak{D}(A)$, and let N and M be the operators, each with domain $\mathfrak{D}_1^*$ and range in $\mathfrak{L}$, such that $Nf=h$ and $Mf=k$ if and only if $A\{f, H^*f\} = \{h, k\}$. Then N and M are linear transformations.*

If S has a totally continuous resolvent and L is an arbitrary bounded self-adjoint transformation in $\mathfrak{L}$, then the boundary condition $Af \, \varepsilon \, \mathfrak{B}(L)$, that is, the requirement $Mf=LNf$, defines a self-adjoint extension of H.

The assertion concerning M and N is evident.

To prove the remainder of the theorem, we denote by L_1 the transformation with domain $\mathfrak{H} \oplus \mathfrak{L}$ which is equal to O in $\mathfrak{H}$ and to L in $\mathfrak{L}$, and assume that S has a totally continuous resolvent. Then L_1 is clearly a bounded self-adjoint transformation in $\mathfrak{H} \oplus \mathfrak{L}$ and it is readily shown that $S - L_1$ is self-adjoint. Moreover, by Lemma 4.5, $S - L_1$ has a totally continuous resolvent; and it can therefore be shown that the range of $S - L_1$ is the orthogonal complement of its manifold of zeros.

Therefore, in particular, the equations $H^*f = f^*$, $Mf - LNf = 0$ have a solution f in $\mathfrak{D}_1^*$ for every f^* in $\mathfrak{H} \ominus \mathfrak{U}$, where $\mathfrak{U}$ is the manifold of elements g of $\mathfrak{D}_1^*$ such that $H^*g = 0$, $Mf - LNf = 0$. But, by definition, every element f of $\mathfrak{D}_1^*$ such that $Mf - LNf = 0$ belongs to the domain of $H(\mathfrak{B}(L))$, where we have reverted to the notation of Definition 1.2; and $H(\mathfrak{B}(L))$ is symmetric, by Theorems 1.5 and 2.7. Moreover, from the result of the preceding paragraph, the range of $H(\mathfrak{B}(L))$ is the orthogonal complement of its manifold of zeros, which we have denoted by $\mathfrak{U}$. Thus $\mathfrak{U}$ and $\mathfrak{H} \ominus \mathfrak{U}$ reduce $H(\mathfrak{B}(L))$; in $\mathfrak{U}$, $H(\mathfrak{B}(L))$ induces the self-adjoint transformation O; and, in $\mathfrak{H} \ominus \mathfrak{U}$, $H(\mathfrak{B}(L))$ induces a transformation whose range is $\mathfrak{H} \ominus \mathfrak{U}$ and which is therefore self-adjoint. Thus $H(\mathfrak{B}(L))$ itself is self-adjoint in $\mathfrak{H}$ and the proof is complete.

The reader may observe that $H(\mathfrak{B}(L))$, being related very simply to a contraction of $S - L_1$, has itself a totally continuous resolvent and thus a pure point spectrum; we refrain, however, from discussing questions of this sort here, but rather reserve them for separate consideration elsewhere.

The hypothesis that S has a totally continuous resolvent which appears in Theorem 4.18 is more restrictive than is, in fact, necessary. For it imposes a restriction on the behavior of the transformation H itself, and this can evidently have no effect on the behavior of the transformation A. However, both the statement and the proof of the more general theorem which is possible are considerably more involved than those of the one which we have given; and, in all of the realizations which we have investigated, Theorem 4.18 applies whenever the more general result does. (In particular, this remark applies to Example 4 of the first chapter; the reduction operator A described there satisfies the hypothesis of Theorem 4.18.)

We have therefore refrained from giving the more general result here and also from investigating further along lines suggested by Theorem 4.18 and the fact already noted that every equivalence class of reduction operators contains, when H has deficiency index (n, n), operators of the sort described in Theorem 3.17: investigations in this direction and other similar ones must be guided to some extent by the nature of possible applications, of which we have at present studied only a few.

7. **Real boundary conditions.** If the transformation H is real with respect to a conjugation J_0 in $\mathfrak{H}$, then H^* is real with respect to J_0 also and H has deficiency index (n, n).[†] In such a situation it is frequently important to determine those maximal symmetric extensions of H which are real with respect to J_0 and which are consequently self-adjoint. For the special case that A is the reduction operator of Theorem 2.9, this problem has a simple solution which is known;[‡] we consider here a more general case.

THEOREM 4.19. *Let H be real with respect to a conjugation J_0 in $\mathfrak{H}$, and let J be the transformation in $\mathfrak{H} \oplus \mathfrak{H}$ which takes $\{f, g\}$ into $\{J_0 f, J_0 g\}$. Then, in accordance with Theorem 2.14, J permutes with Q in $\mathfrak{B}^+ + \mathfrak{B}^-$ and $\mathfrak{B}^*$ is real with respect to J.*

Let A be a reduction operator for H^*, let $\mathfrak{D}(A)$ be real with respect to J, and let J_1 be a conjugation in the range-space $\mathfrak{M}$ of A such that $AJ\{f, H^*f\} = J_1 A\{f, H^*f\}$ for every $\{f, H^*f\}$ in $\mathfrak{D}(A)$. Then J_1 permutes with W.

Let V be an arbitrary closed isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that the boundary condition $Af \varepsilon \Re(I - V)$ is nondegenerate. Then $H(V)$ is real with respect to J_0 if and only if $\Re(I - V)$ is real with respect to J_1 or, equivalently, if and only if $V \equiv J_1 V^{-1} J_1$. Thus, if $V \equiv J_1 V^{-1} J_1$, then $\tilde{H}(V)$ has deficiency index (n, n); consequently $H(V)$ is self-adjoint if it is maximal symmetric, essentially self-adjoint if $\tilde{H}(V)$ is maximal symmetric.

As we have noted in the statement of the theorem, the assertions of the first paragraph are consequences of Theorem 2.14.

We now prove that J_1 permutes with W. Let $\{f, H^*f\}$, $\{g, H^*g\}$ be arbitrary elements of $\mathfrak{D}(A_1)$. Then

$$(J_1 A_1\{f, H^*f\}, J_1 W A_1\{g, H^*g\}) = \overline{(A_1\{f, H^*f\}, W A_1\{g, H^*g\})}$$

$$= -\overline{(\{f, H^*f\}, Q\{g, H^*g\})},$$

where the last inner product is formed in $\mathfrak{H} \oplus \mathfrak{H}$. But

$$\overline{(\{f, H^*f\}, Q\{g, H^*g\})} = (J\{f, H^*f\}, QJ\{g, H^*g\}),$$

since J permutes with Q; and

$$(J\{f, H^*f\}, QJ\{g, H^*g\}) = -(A_1 J\{f, H^*f\}, W A_1 J\{g, H^*g\})$$

$$= -(J_1 A_1\{f, H^*f\}, W J_1 A_1\{g, H^*g\}).$$

Thus we have

$$(J_1 A_1\{f, H^*f\}, J_1 W A\{g, H^*g\}) = (J_1 A_1\{f, H^*f\}, W J_1 A\{g, H^*g\})$$

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† Stone, Theorems 9.13 and 9.14.
‡ Stone, pp. 362–364.

for all $\{f, H^*f\}$, $\{g, H^*g\}$ in $\mathfrak{D}(A_1)$. But $J_1A_1\mathfrak{D}(A_1)$ is dense in $\mathfrak{M}$ and hence

$$J_1WA_1\{g, H^*g\} = WJ_1A_1\{g, H^*g\}$$

for all $\{g, H^*g\}$ in $\mathfrak{D}(A)$. Therefore $J_1W = WJ_1$ on $\mathfrak{R}(A)$. But $\overline{\mathfrak{R}}(A) = \mathfrak{M}$, while J_1W and WJ_1 are continuous. Consequently $J_1W \equiv WJ_1$, as we wished to prove.

Now let V be the transformation described in the last paragraph of the theorem. We note first that $H(V)$ is real with respect to J_0 if and only if its graph is real with respect to J; furthermore, $\mathfrak{B}$ is clearly real with respect to J, since H is real with respect to J_0. Hence, setting

$$\mathfrak{B}_1(H(V)) = \mathfrak{B}(H(V)) \cdot (\mathfrak{B}^+ + \mathfrak{B}^-),$$

we conclude that $H(V)$ is real with respect to J_0 if and only if $\mathfrak{B}_1(H(V))$ is real with respect to J; and thus, by Theorem 2.12, if and only if $J\mathfrak{B}_1(H(V)) = \mathfrak{B}_1(H(V))$.

We shall now show that the latter equation holds if and only if $\mathfrak{R}(I - V)$ is real with respect to J_1, beginning with the observation that $J_1A_1\mathfrak{B}_1(H(V)) = A_1J\mathfrak{B}_1(H(V))$. Hence, if $J\mathfrak{B}_1(H(V)) = \mathfrak{B}_1(H(V))$, we must have $J_1A_1\mathfrak{B}_1(H(V)) = A_1\mathfrak{B}_1(H(V))$, and the latter implies that $A_1\mathfrak{B}_1(H(V))$ is real with respect to J_1, by Theorem 2.12. Hence, since $\mathfrak{R} = \mathfrak{R}(I - V)$ is the closure of $A_1\mathfrak{B}_1(H(V))$, $\mathfrak{R}$ is real with respect to J_1, again by Theorem 2.12. On the other hand, suppose $\mathfrak{R} = \mathfrak{R}(I - V)$ is real with respect to J_1. Then, since $J_1A\{f, H^*f\}$ is in $\mathfrak{R}(A)$ for every $\{f, H^*f\}$ in $\mathfrak{D}(A)$, $\mathfrak{R}(A)$ is real with respect to J_1 and therefore $\mathfrak{R} \cdot \mathfrak{R}(A)$ is also. But

$$A_1^{-1}[\mathfrak{R} \cdot \mathfrak{R}(A)] = \mathfrak{B}_1(H(V))$$

and

$$A_1^{-1}[\mathfrak{R} \cdot \mathfrak{R}(A)] = A_1^{-1}J_1[\mathfrak{R} \cdot \mathfrak{R}(A)] = JA_1^{-1}[\mathfrak{R} \cdot \mathfrak{R}(A)].$$

Consequently $J\mathfrak{B}_1(H(V)) = \mathfrak{B}_1(H(V))$.

Thus, combining the results of the two preceding paragraphs, we conclude that $H(V)$ is real with respect to J_0 if and only if $\mathfrak{R} = \mathfrak{R}(I - V)$ is real with respect to J_1; and, as we have already shown that W permutes with J_1, the equivalence of the condition that $\mathfrak{R}$ be real with respect to J and the condition $V \equiv JV^{-1}J$ follows from Theorem 2.13.

We are thus left to prove only the assertions of the final sentence in the theorem. This, however, requires only the observation that $\tilde{H}(V)$ is evidently real with respect to J_0 when $H(V)$ is; the statements in question then follow from the results already established and Theorem 9.14 of the book of Stone previously cited.

We point out that the hypothesis of the existence of the conjugation J_1 is necessary and does not follow from the reality of H and H^*; the hypothesis is, however, in accord with the situation which exists in the theory of differential operators. If such an assumption is not introduced, apparently little of interest can be said concerning those boundary conditions which define real extensions of H.

Theorem 4.19 has, of course, a special interpretation for the boundary conditions considered in Theorem 4.18; we leave the formulation of this to the reader.

8. Formulation of boundary conditions. In developing our theory we have found it necessary only to use the representations $Af \varepsilon \mathfrak{N}$, or $Af \varepsilon \mathfrak{R}(I-V)$ for the boundary conditions under consideration. In the applications of the theory, however, different statements of the conditions may be found convenient. We now discuss briefly some of these. We consider an arbitrary nondegenerate boundary condition $Af \varepsilon \mathfrak{N}$, $\mathfrak{N}$ being closed linear W-symmetric.

If $\{\phi_n\}$, $n=1, 2,\cdots$, is an arbitrary sequence which determines the closed linear manifold $\mathfrak{M} \ominus \mathfrak{N}$, the condition $Af \varepsilon \mathfrak{N}$ is evidently equivalent to the conditions $(Af, \phi_n)=0$, $n=1, 2, \cdots$. Now let $\{\psi_m\}$ be a complete orthonormal set in $\mathfrak{M}$, and let $\phi_n=\sum_m \bar{a}_{mn}\psi_m$, $n=1, 2, \cdots$; $Af=\sum_m b_m(f)\psi_m$, for each Af in $\mathfrak{N}$. Then $(Af, \phi_n)=\sum_m a_{mn}b_m(f)$. Accordingly, since the sequences $\{b_m(f)\}$ are evidently in one-to-one correspondence with the elements Af in $\mathfrak{N}$, the condition $Af \varepsilon \mathfrak{N}$ is equivalent to the conditions $\sum_m a_{mn}b_m(f)=0$, $n=1, 2, \cdots$. If $\mathfrak{M}$ is a Hilbert space, there is, in general, an infinite number of these equations; if $\mathfrak{M}$ is unitary, there is only a finite number.

We may point out that the form of representation of the boundary conditions generally used in the theory of ordinary differential equations is precisely of the sort just described.

Another simple representation of the boundary conditions is a parametric one. If $\mathfrak{N}$ is W-symmetric in $\mathfrak{M}$ and V is the isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $\mathfrak{N}=\mathfrak{R}(I-V)$, we may regard the manifold $\mathfrak{D}(V)$ as a space of parameters and define the domain of $H(V)$ as that set of elements f of $\mathfrak{D}^*$ such that Af is defined and satisfies the equation $Af=h-Vh$ for some h in $\mathfrak{D}(V)$. In this connection, we observe that we are at liberty to choose coordinates in $\mathfrak{M}^+$ and in $\mathfrak{M}$ as we please.

Finally, we consider the special case where A is a reduction operator of the kind described in Theorem 3.17. Here, the range of A is in a space $\mathfrak{L} \oplus \mathfrak{L}$, and we can set $Af=\{Nf, Mf\}$, where Nf and Mf have the same meanings as in Theorem 4.18. In this case, if $\mathfrak{N}$ is a closed W-symmetric manifold in $\mathfrak{L} \oplus \mathfrak{L}$, then by Theorem 2.10, there exists a uniquely determined closed linear

manifold $\mathfrak{U}$ in $\mathfrak{L}$ and a uniquely determined Hermitian transformation L in $\mathfrak{L} \ominus \mathfrak{U}$, such that the condition $Af \; \varepsilon \; \mathfrak{N}$ is equivalent to the condition $Mf = LNf + h$, where h is in $\mathfrak{U}$. In particular, the boundary conditions of potential theory are of this form. Moreover, M. Morse has found essentially this form of representation useful in connection with comparison theorems for systems of ordinary differential equations of the second order.[†] In the terminology of Morse, the space $\mathfrak{L} \ominus \mathfrak{U}$ is the "accessory end-plane," and the form (k, Lk) the "accessory end-form."

9. Homogeneous boundary value problems. The theory which we have developed (particularly the results of Chapter IV, §4) provides information concerning the solvability of the equation $H^*f - \lambda f = f^*$ under a restriction $Af \; \varepsilon \; \mathfrak{N}$, where $\mathfrak{N}$ is a W-symmetric manifold in $\mathfrak{M}$. For example, if the boundary condition $Af \; \varepsilon \; \mathfrak{N}$ defines a self-adjoint extension of H, and if f^* is an arbitrary element of $\mathfrak{H}$, λ an arbitrary complex such that $\Im(\lambda) \neq 0$, then there exists one and only one element f of $\mathfrak{D}^*$ such that $H^*f - \lambda f = f^*$, $Af \; \varepsilon \; \mathfrak{N}$.

The theory of differential equations suggests quite naturally that we should consider, as well as questions of the above sort, problems of the following kind: given a linear manifold $\mathfrak{N}$ in $\mathfrak{M}$, to determine an element f of $\mathfrak{D}^*$ such that $H^*f - \lambda f = 0$, $E_\mathfrak{N} Af = h$, where λ is a complex number and h a preassigned element of $\mathfrak{N}$. We bring this paper to its end with a few observations concerning such problems.

THEOREM 4.20. *Let V be an arbitrary closed isometric transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ such that $H(V)$ is a maximal symmetric extension of H. Let $\mathfrak{N} = \Re(I - V)$, and let λ belong to the resolvent set of $H(V)$. Then, for every element h of an everywhere dense linear manifold in $\mathfrak{P} = \mathfrak{M} \ominus \mathfrak{N}$, there exists a solution f of the equation $H^*f - \lambda f = 0$ such that Af is defined and $E_\mathfrak{P} Af = h$; if $\mathfrak{P}$ belongs to the range of A in particular, if A is bounded, then a solution f exists for every h in $\mathfrak{P}$. Moreover, f is uniquely determined by h.*

To prove this theorem, we note first that if λ is in the resolvent set of $H(V)$, every element g of $\mathfrak{D}^*$ has a unique resolution of the form $g = f_1 + f$, where f_1 belongs to $\mathfrak{D}(H(V))$ and f is a solution of the equation $H^*f - \lambda f = 0$; to demonstrate this, we have only to set

$$f_1 = (H(V) - \lambda I)^{-1}(H^* - \lambda I)g, \qquad f = g - f_1.$$

In particular, every element g such that Ag is defined can be written in this form and, for such an element, since Af_1 is defined by hypothesis, Af is also. Furthermore, for every such g, $E_\mathfrak{P} Ag = E_\mathfrak{P} Af$, since Af_1 is in $\mathfrak{N} = \mathfrak{M} \ominus \mathfrak{P}$.

† M. Morse, *The Calculus of Variations in the Large*, American Mathematical Society Colloquium Publications, vol. 18, New York, 1934, chap. 4, especially pp. 83–89.

Therefore, the system $H^*f - \lambda f = 0$, $E_\mathfrak{B} A f = h$ has a solution f for every element h in $E_\mathfrak{B} \Re(A)$. But $\Re(A)$ is dense in $\mathfrak{M}$ and $E_\mathfrak{B} \Re(A)$ is therefore a dense linear manifold in $\mathfrak{P}$. Moreover, if $\mathfrak{P}$ belongs to $\Re(A)$ (and this is necessarily true when A is bounded) we have $E_\mathfrak{B} \Re(A) = \mathfrak{P}$.

Thus it remains only to prove that the solution f is unique. Let us suppose that f_1 and f_2 are two solutions for the same element h of $\mathfrak{P}$. Then $A(f_1 - f_2)$ belongs to $\mathfrak{M} \ominus \mathfrak{P} = \Re$. Thus $f_1 - f_2$ is in the domain of $H(V)$. But

$$H(V)(f_1 - f_2) - \lambda(f_1 - f_2) = 0$$

and λ is in the resolvent set of $H(V)$. Hence $f_1 = f_2$, and f is uniquely determined by h, as we wished to show.

It is interesting to observe that suitably modified formulations of the Dirichlet and Neumann problems of potential theory can be described in terms of the abstract problem considered in Theorem 4.20. Furthermore, the so-called boundary value problem of the third kind is closely connected with a problem of this sort, as the following theorem suggests:

THEOREM 4.21. *Let A be a reduction operator of the sort described in Theorem 3.17 with range-space $\mathfrak{L} \oplus \mathfrak{L}$, and let M and N have the same meanings as in Theorem 4.18. Let L be a bounded self-adjoint transformation in $\mathfrak{L}$, and let the boundary condition $Af \, \varepsilon \, \mathfrak{B}(L)$ be nondegenerate and define a maximal symmetric extension of H. Then if λ is in the resolvent set of the extension so defined, the system*

$$H^*f - \lambda f = 0, \qquad LNf - Mf = k$$

has a unique solution f for every element k of an everywhere dense linear manifold in $\mathfrak{L}$.

According to Theorem 4.20, the system

$$H^*f - \lambda f = 0, \qquad E_\mathfrak{B}\{Nf, Mf\} = \{r, s\}$$

has a unique solution f for every element $\{r, s\}$ in a dense linear subset of $\mathfrak{P}$, where $\mathfrak{P} = (\mathfrak{L} \oplus \mathfrak{L}) \ominus \mathfrak{B}(L)$. Since L is self-adjoint, in the second equation above, we can set $r = Lh$, $s = -h$. Then $\{Nf, Mf\} = \{Lh, -h\} + \{t, Lt\}$, where t is some element of $\mathfrak{L}$. Thus

$$LNf = L^2h + Lt, \quad Mf = -h + Lt.$$

Therefore $LNf - Mf = (L^2 + I)h$. Thus the solution f of the system

$$H^*f - \lambda f = 0, \qquad E_\mathfrak{B}\{Nf, Mf\} = \{Lh, -h\}$$

is also a solution of the system

$$H^*f - \lambda f = 0, \qquad LNf - Mf = k,$$

with $k = (L^2+I)h$. Conversely, it is easily shown that a solution f of the second system determines a solution of the first, again with $k = (L^2+I)h$. Therefore, since the equation $k = (L^2+I)h$ is readily shown to determine a one-to-one linear bicontinuous correspondence between the elements k of $\mathfrak{L}$ and the elements $\{Lh, -h\}$ of $\mathfrak{P}$, the theorem follows.

If, in particular, the transformation S in $\mathfrak{H} \oplus \mathfrak{L}$, associated with A by Theorem 3.17, satisfies the hypothesis of the second paragraph of Theorem 4.18, we are able to attack questions of the sort considered in Theorem 4.21 by methods similar to those used in proving Theorem 4.18.

THEOREM 4.22. *Let A, $\mathfrak{L}$, M, N, and S have the same meanings as in Theorem 4.18, and let S have a totally continuous resolvent. Let L be an arbitrary bounded self-adjoint transformation in $\mathfrak{L}$, and let T be the extension of H determined by the boundary condition $Af \in \mathfrak{B}(L)$; that is to say, by the condition $LNf = Mf$. Let λ be a real number, and let $(T - \lambda I)^{-1}$ exist. Then the system*

$$H^*f - \lambda f = 0, \qquad Mf - LNf = h$$

has one and only one solution f for every h in $\mathfrak{L}$.

In view of Theorem 1.3, it is sufficient to prove the theorem for the case $\lambda = 0$.

Again we introduce the transformation L_1 used in the proof of Theorem 4.18. Then, since $S - L_1$ has a totally continuous resolvent, it is a simple task to show that the origin is either in its point spectrum or its resolvent set; we omit the details. Since T^{-1} exists, it is clear that the origin cannot belong to the point spectrum of $S - L_1$; therefore, it belongs to the resolvent set. Accordingly, the equation $(S - L_1)\{f, k\} = \{0, h\}$ has a unique solution $\{f, k\}$ in $\mathfrak{H} \oplus \mathfrak{L}$ for every h in $\mathfrak{L}$. Consequently, since f is then a solution of the system $H^*f = 0$, $Mf - LNf = h$, the proof is complete.

To conclude, we suggest two simple generalizations, in different directions, of Theorem 4.22. First, if λ is a real number in the point spectrum of the transformation T, and $\mathfrak{P} = N\mathfrak{U}$, where $\mathfrak{U}$ is the manifold of zeros of T, it can be shown that $\mathfrak{P}$ has a finite dimension number and that the system

$$H^*f - \lambda f = 0, \qquad Mf - LNf = h$$

has a solution f for every h in $\mathfrak{L} \ominus \mathfrak{P}$, but the solution is not unique. Second, Theorem 4.22 can be extended to cover the case that λ is not real, by taking account of the fact that the transformation in $\mathfrak{H} \oplus \mathfrak{L}$ which takes $\{f, Nf\}$ into $\{H^*f - \lambda f, Mf\}$ is normal. We leave to the reader the proof of these assertions.

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